# A Smooth Representation of Finite Dimensional Hilbert Space

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A representation of a finite dimensional Hilbert space using smooth wave functions is shown. A continuum of states, $|x\rangle$ for $x \in \mathbb{R}$ is defined such that the inner product can be expressed either as a finite sum or as an integral. The orthogonality condition can be expressed using either a Kronecker or Dirac delta, where the Dirac delta is a smooth function. Operators do not generally have an integral form. The representation has potential applications in quantum theory, particularly with regard to the treatment of divergences in quantum electrodynamics.

## 1. Introduction

It is an inconvenience for treatments in quantum mechanics that states of exact position and momentum are not vectors in Hilbert space. They can be treated with rigged Hilbert space, but a proper justification based on nuclear space is notoriously difficult. Further issues arise because there is no physical justification for states with exact position and momentum, and because perturbative quantum field theory uses products of distributions which do not always exist, resulting in the well known divergences associated with certain loops in Feynman diagrams.

This paper considers a representation of finite dimensional Hilbert space in which vectors are smooth functions, and which contains vectors representing states of exact position and momentum. The inner product can be represented either as a finite sum or as an integral on a real interval, and the orthogonality condition can be equivalently expressed using either a Kronecker or Dirac delta, where the Dirac delta is a smooth function.

I will work in one dimension. The generalisation to three dimensions is straightforward. Scaling and normalisation is chosen to admit a simple transition to the integral formulation. I will use Dirac ket notation for vectors, for reasons of elegance and clarity. Members of the dual space are bras, and can also be regarded as operators.

## 2. The Smooth Transform

For some $\nu \in \mathbb{N}$. Let $\mathscr{D} = \{x \in \chi\mathbb{Z}, x = -\chi(\nu+1), \ldots, \chi\nu\}$. $\mathscr{D}$ will be called positon space. Let $\mathbb{H}$ be a $2\nu$ dimensional Hilbert space, with basis $\mathscr{B} = \{|x\rangle, x = -\chi(\nu-1), \ldots, \chi\nu\}$ and inner product

$$\langle g|f\rangle = \chi \sum_{x \in \mathscr{D}} \langle g|x\rangle\langle x|f\rangle,$$

where the basis states have the normalisation

$$\langle x | y \rangle = \chi^{-1} \delta_{xy} .$$

**Definition:** For $p \in \mathscr{M} \equiv \dfrac{\pi}{\nu\chi}[-\nu, \nu] \subset \mathbb{R}$

$$|p\rangle = \left(\tfrac{1}{2\pi}\right)^{1/2} \chi \sum_{x \in \mathscr{D}} e^{ixp} |x\rangle ,$$

$\mathscr{M}$ will be called momentum space. It is a standard result that the subset

$$\{|p\rangle, p \in \mathscr{M}_{\mathscr{D}} = \mathscr{M} \cap \chi_p \mathbb{N}, \chi_p = \pi/(\chi\nu)\} ,$$

is an orthogonal basis for $\mathbb{H}$;

for $p, q \in \mathscr{M}_{\mathscr{D}}$, $\langle p | q \rangle = \chi_p^{-1} \delta_{pq}$

The identity operator $1 : \mathbb{H} \to \mathbb{H}$ can be resolved:

$$1 = \chi_p \sum_{p \in \mathscr{M}_{\mathscr{D}}} |p\rangle\langle p| .$$

**Definition:** For the ket, $|f\rangle \in \mathbb{H}$, *the momentum space wave function*, $F : \mathscr{M} \to \mathbb{C}$, is given by

$$p \to F(p) = \langle p | f \rangle .$$

In particular, for the basis ket $|z\rangle \in \mathscr{B}$, $F_z : \mathscr{M} \to \mathbb{C}$ is a smooth function of $p$

$$p \to F_z(p) = \langle p | z \rangle = \left(\tfrac{1}{2\pi}\right)^{1/2} e^{-izp}$$

For $x, y \in \mathscr{D}$,

$$\int_{\mathscr{M}} dp \langle x | p \rangle\langle p | y \rangle = \frac{1}{2\pi} \int_{\mathscr{M}} dp e^{-iyp} \, e^{ixp} = \chi^{-1} \delta_{xy} = \langle x | y \rangle .$$

Thus, Fourier inversion holds using an integral on $\mathscr{M}$; for any $|f\rangle \in \mathbb{H}$,

$$\int_{\mathscr{M}} dp \langle x | p \rangle\langle p | f \rangle = \int_{\mathscr{M}} dp \chi \sum_{y \in \mathscr{D}} \langle x | p \rangle\langle p | y \rangle\langle y | f \rangle = \langle x | f \rangle .$$

So, we can identify the sum with an integral:

$$1 = \chi_p \sum_{p \in \mathscr{M}_{\mathscr{D}}} |p\rangle\langle p| \equiv \int_{\mathscr{M}} dp |p\rangle\langle p|$$

Then for any $|f\rangle \in \mathbb{H}$, $q \in \mathscr{M}$

$$\langle q | f \rangle = \chi_p \sum_{p \in \mathscr{M}_{\mathscr{D}}} \langle q | p \rangle\langle p | f \rangle \equiv \int_{\mathscr{M}} dp \langle q | p \rangle\langle p | f \rangle .$$

and for any $p, q \in \mathscr{M}$

$$\langle q | p \rangle = \delta(p - q)$$

where the Dirac delta is defined as a smooth function:

$$\delta(p - q) = \frac{1}{2\pi} \chi \sum_{x \in \mathscr{D}} e^{ix(p-q)} .$$

## 3. Smooth Wave Functions

Let $\mathscr{C} \equiv [-\chi\nu, \chi\nu] \subset \mathbb{R}$ be the real interval containing $\mathscr{D}$.

**Definition:** For any $x \in \mathscr{C}$ we may define the ket

$$|x\rangle = \chi_p \sum_{p \in \mathscr{M}_{\mathscr{D}}} |p\rangle\langle p|x\rangle = \int_{\mathscr{M}} dp|p\rangle\langle p|x\rangle$$

**Definition:** The wave function for $|f\rangle \in \mathbb{H}$ is the smooth function $f{:}\mathscr{C} \to \mathbb{C}$.

$$x \to f(x) = \chi_p \sum_{p \in \mathscr{M}_{\mathscr{D}}} \langle x|p\rangle\langle p|f\rangle = \int_{\mathscr{M}} dp\langle z|p\rangle\langle p|f\rangle$$

The wave function for $|z\rangle$, $z \in \mathscr{C}$ is, for $x \in \mathscr{C}$,

$$x \to f_z(x) = \int_{\mathscr{M}} dp\langle x|p\rangle\langle p|z\rangle = \frac{1}{2\pi}\int_{\mathscr{M}} dpe^{ixp}e^{-izp}$$

Thus

$$f_z(x) = \frac{1}{2\pi}\int_{\mathscr{M}} dpe^{i(x-z)p}$$

$$= \begin{cases} \chi^{-1} & \text{if } x = z \\ \dfrac{e^{i(x-z)\pi/\chi} - e^{-i(x-z)\pi/\chi}}{2\pi i(x-z)} & \text{otherwise.} \end{cases}$$

Thus, for $x, z \in \mathscr{D}$

$$f_z(x) = \chi^{-1}\delta_{xz} = \langle x|z\rangle.$$

There is thus a one-one correspondence between the wave functions, $f_z(x)$, and basis kets, $|z\rangle$. By linearity, for any $|f\rangle \in \mathbb{H}$ the coefficients of $|f\rangle$ in the expansion in terms of $\mathscr{B}$, are given by the restriction of the wave function to $\mathscr{D}$. Then for $p, q \in \mathscr{M}_{\mathscr{D}}$

$$\int_{\mathscr{C}} dx\langle p|x\rangle\langle x|q\rangle = \frac{1}{2\pi}\int_{\mathscr{C}} dxe^{-ix(p-q)} = \chi_p^{-1}\delta_{pq} = \langle p|q\rangle$$

So, by linearity, we can identify the sum over discrete coordinates with an integral, such that the identity operator $1{:}\mathbb{H} \to \mathbb{H}$ can be written

$$1 = \chi_p \sum_{x \in \mathscr{D}} |x\rangle\langle x| \equiv \int_{\mathscr{C}} dx|x\rangle\langle x|$$

Then, for any $|f\rangle \in \mathbb{H}$, $y \in \mathscr{C}$

$$\langle y|f\rangle = \chi \sum_{x \in \mathscr{D}} \langle y|x\rangle\langle x|f\rangle \equiv \int_{\mathscr{C}} dx\langle y|x\rangle\langle x|f\rangle.$$

and for any $x, y \in \mathscr{C}$

$$\langle y|x\rangle = \delta(x-y),$$

where the Dirac delta is defined as a smooth function:

$$\delta(x-y) = \chi \sum_{p \in \mathscr{M}_{\mathscr{D}}} e^{i(x-y)p} = \int_{\mathscr{M}} dpe^{i(x-y)p}.$$

## 4. The Canonical Commutation Relation

It has been seen that, with the above definitions, wave functions are a representation of a finite dimensional Hilbert space, and that integrals are interchangeable with sums in the inner product, and the Dirac delta is interchangeable with the Kronecker delta, up to normalisation. It is possible to define derivatives of vectors. For example, we may define the momentum operator $P = -i\partial\colon \mathbb{H} \to \mathbb{H}$,

$$P\colon |f\rangle \to -\int_{\mathscr{C}} dx|x\rangle i\partial\langle x|f\rangle = -\int_{\mathscr{C}} dx|x\rangle i\partial\chi_p \sum_{p \in \mathscr{M}_{\mathscr{D}}} \langle x|p\rangle\langle p|f\rangle$$

$$= \chi_p \sum_{p \in \mathscr{M}_{\mathscr{D}}} |p\rangle p\langle p|f\rangle \in \mathbb{H}.$$

Clearly $P$ is Hermitian. Note that

$$P \neq \int_{\mathscr{M}} dp|p\rangle p\langle p|f\rangle.$$

The position operator, $X\colon \mathbb{H} \to \mathbb{H}$, is given by

$$X|f\rangle = \chi \sum_{x \in \mathscr{D}} |x\rangle x\langle x|f\rangle.$$

It is not possible to differentiate under the sum in the product $PX$. From the property that the trace of a commutator in finite dimensional Hilbert space vanishes,

$$\mathrm{Tr}([X, P]) = 0,$$

it is seen that we do not have the canonical commutation relation,

$$[X, P] \neq i.$$

If we formally define $\tilde{X}$ by

$$\tilde{X}|f\rangle = \int_{\mathscr{C}} dx|x\rangle x\langle x|f\rangle$$

Then

$$P\tilde{X}|f\rangle = -\int_{\mathscr{C}} dx|x\rangle i\langle x|f\rangle - \int_{\mathscr{C}} dx|x\rangle xi\partial\langle x|f\rangle = -i - \tilde{X}P|f\rangle.$$

So,

$$[\tilde{X}, P] = i,$$

and we conclude that $X \neq \tilde{X}$ and that $\tilde{X}|f\rangle \notin \mathbb{H}$. The canonical commutation relation can be obtained by forming the union of the sets of wave functions for all $\chi$ and $\nu$.

## 5. Application

It can be argued that, since physical measurement has finite range and resolution and since the inner product in quantum theory is related to measurement results, it is appropriate to use finite dimensional space. The use of a representation of finite dimensional Hilbert space using smooth functions makes it possible to define differential operators. The author has used this formulation in a treatment of quantum electrodynamics (paper in preparation), in which field operators are functions, not distributions, and the existence of an upper bound on momentum leads naturally to the energy cut-off used in the treatment of loop divergences. Consistency requires that physical predictions are independent of $\chi$ and $\nu$ (for sufficiently large $\chi^{-1}$ and $\nu$).

Since physical momentum is bounded by considerations of energy conservation, momentum space wave functions representing physical states have bounded support. Provided that $\chi$ is sufficiently small, the bound, $\pi/\chi$, on momentum has no physical import. A form of covariance can be preserved because discrete coordinates, $\mathscr{C}^3$ are naturally determined from the measurement apparatus used to define the reference frame.

In a change of reference frame, the discrete coordinates and inner product appropriate to one apparatus is naturally replaced with the discrete coordinates and inner product appropriate to another.