

Relational Quantum Gravity I: Conceptual and Mathematical Foundations

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ABSTRACT

Context. The well known problems of interpretation of quantum mechanics mean that physical law lacks a physical basis.

Aims. To derive the formalism of quantum mechanics from assertions about the world with clear physical meaning.

Methods. Kets are defined as formal conditional clauses referring to measurements in a formal language and have a natural structure as a vector space. Addition is logical disjunction. The dual space consists of consequent clauses, such that the inner product is a set of statements in the subjunctive mood. The probability interpretation gives truth values for corresponding future tense statements when the initial state is prepared and the final state is to be measured.

Results. The mathematical structure of quantum mechanics is formulated in terms of discrete measurement results at finite level of accuracy and does not depend on an assumption of a background spacetime continuum. Discrete position functions are uniquely embedded into smooth wave functions in such a way that differential operators are defined. Using finite dimensional Hilbert space, a continuum of states, $|x\rangle$ for $x \in \mathbb{R}^3$, is defined such that the inner product can be expressed either as a finite sum or as an integral. Operators do not in general have an integral form. The Schrödinger equation is shown from the requirements of the probability interpretation.

Key Words: Foundations of quantum mechanics; Quantum logic; Quantum gravity; Fourier analysis

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1 Introduction

1.1 Objectives

Carlo Rovelli (1996) describes the purpose of Relational Quantum Mechanics: “... to do for the formalism of quantum mechanics what Einstein did for the Lorentz transformations: i. Find a set of simple assertions about the world, with clear physical meaning, that we know are experimentally true (postulates); ii. Analyze these postulates, and show that from their conjunction it follows that certain common assumptions about the world are incorrect; iii. Derive the full formalism of quantum mechanics from these postulates. I expect that if this program could be completed, we would at long last begin to agree that we have understood quantum mechanics”.

To say that we have completed such a program it is not sufficient to present a consistent mathematical structure giving correct predictions. A mathematical model is defined

from its axioms, and we require that the axioms are physically sensible in addition to being logically consistent and empirically valid. The defining axioms for the mathematical structure described here will be termed postulates and definitions; postulates are intended to contain empirical assertions about the world, while definitions are purely semantic.

The purpose of this paper is to develop a formulation of quantum mechanics from observationally true postulates. Rather than start with the mathematical theory and try to interpret it, I adopt a specific (Dirac-Von Neumann, q.v. Bub, 1997) interpretation and seek to produce the mathematical structure appropriate to it. The result is essentially relativistic quantum mechanics, but with subtle and sometimes important differences. Of fundamental importance, and in contrast to standard quantum theory, the model is background-free in the sense that the physical metric is determined from measurement results. Hilbert space is finite dimensional, but a continuum of states, $|x\rangle$ for $x \in \mathbb{R}^3$, is defined using linear combinations of basis states (this may be compared with the definition of a sphere as a continuous surface in three dimensions). Other changes from standard quantum theory include discreteness of measurement results, and quantum covariance in lieu of manifest covariance. A wave equation governing time evolution is not assumed as a postulate, but is established from probabilistic considerations. Neither momentum space nor the commutation relations are assumed, since they are contained in the mathematical structure of Hilbert space. There is no assumption of a Lagrangian, and, Poincaré invariance is not used in relativistic qed. It is essential to the development in subsequent papers to recognise that the wave function is not defined on space, but in Hilbert space defined at the position of measurement.

Relational quantum gravity is based on the observation that when there is no means to define the coordinate of a particle, quantum effects appear. The interpretation follows Dirac (1958) and Von Neumann (1955), has its origins in the Copenhagen interpretation as discussed by Heisenberg (1962), and shares much with modern views such as Mermin (1998), Adami and Cerf (1995), and Rovelli (1996). As in the Copenhagen interpretation matter has an unknown but real behaviour which is not directly described by quantum mechanics. By giving a probability for each outcome, the state describes not what is but our knowledge of what might happen in measurement; quantum theory is essentially a theory of probabilistic relationships between measurement results, not a model of physical processes between measurements.

The Dirac-Von Neumann interpretation should not be conflated with the Copenhagen interpretation, since Copenhagen invokes some notion of complementarity which is absent in Dirac-Von Neumann. The interpretation here follows Dirac and Von Neumann, but goes further than either. For example, Dirac stated what cannot be said of quantum particles, but not what can be said, but here a particle is defined as a physical entity in the absence of spacetime background. Von Neuman described quantum logic is a language which tells us what can be discovered from measurement but he did not translate the propositions of quantum logic into English. Here they are abstracted them from the formal statement of sentences in ordinary language. This is not simply a restatement of a standard interpretation, but is a *demonstration* that statements of formal language describing possible measurement results obey the mathematical structure of Hilbert space.

The formalism given here will be used in subsequent papers as a basis for a construction of qed (Francis 2009a, hereafter RQG II) and as the basis for a reconciliation between quantum theory and general relativity (Francis 2009b, hereafter RQG III). Feynman diagrams will have a physical meaning and divergence issues will be resolved. Maxwell's equations and the Lorentz force law will be derived from the interaction between photons and electrons. The treatment of expansion will lead to testable predictions and has implications for missing mass, the cosmological constant, lensing, galaxy rotation curves and the anomalous Pioneer blue shift (Francis 2009c, hereafter RQG IV). A specific test on over 20 000 local stars with accurate Hipparcos parallaxes and spectrographic radial velocities confirms the existence of a signature for the prediction that the apparent flattening of galaxy rotation curves is spectrographic not kinematic and has an origin in cosmological expansion. The resolution of quantum paradoxes will be discussed in Francis (2009d, hereafter RQG V).

It is essential to the development that, because Hilbert space consists of clauses in formal language not wave functions on space, it is defined *at the position of measurement*, not in spacetime. Since the transform used to define momentum space is simply a change of basis of Hilbert space, global Poincaré invariance is not required. The integral used to define a transform necessarily refers to states in a Hilbert space defined at particular position, not to either flat or curved space.

The *philosophical* position that interpretation is not physics has never been justified — indeed, it cannot be justified because a philosophy which rejects philosophy also rejects itself. Unification refutes any such philosophy by showing that empirical results in cosmology follow from the correct interpretation of quantum mechanics.

The treatment given here neglects spin. The inclusion of spin raises additional issues concerning the interpretation of the projection postulate. By ignoring spin these issues do not arise. The present treatment is extended in RQG II, where it is recognised that spin is a required property of particles in a relativistic quantum theory. Measurement issues concerning the reasons for the projection postulate as applied to spin depend upon the physical processes involved in measurement, and can only be resolved after considering quantum electrodynamics as a theory of interactions between particles.

1.2 Relationism

Relationism is the principle that, since a measurement of distance is a comparison between the matter (and radiation) being measured and the matter (and radiation) it is measured against, only relative distances should appear at a fundamental level in physical theory. Although the mathematical formulation of physical law has depended on an assumption of space, or more recently spacetime, imbued with mathematical properties, the Cartesian relationist view continues to hold intellectual appeal and, as described by Dieks (2001), there is some reason, both within the foundations of quantum mechanics and in relativity, for thinking that the correct way to formulate physical theory would be to describe spacetime as a collection of frame-dependent sets of potential measurement results, rather than as a background into which matter is placed in the manner of Newtonian space. In recent years relationism has been used by Smolin (1997), Rovelli (2000) and others as motivation for work on background-free theories such as spin networks,

and has been suggested as basis for understanding quantum mechanics (Rovelli 1996) and quantum gravity (Poulin 2006).

Relativity of motion is often stated, ‘*you cannot say how something is moving unless you say how it is moving relative to other matter*’. The relationist view also requires relativity of position; ‘*you cannot say where something is unless you say where it is relative to other matter*’. Relationism is also suggested by the orthodox, or Dirac-Von Neumann, interpretation of quantum mechanics, that it only makes sense to talk of measured values when a measurement is actually done. “*In the general case we cannot speak of an observable having a value for a particular state, but we can.... speak of the probability of its having a specified value for the state, meaning the probability of this specified value being obtained when one makes a measurement of the observable*” (Dirac, 1958). We may infer from Dirac’s words that a precise value of position only exists when a measurement of position is performed, so that we can only talk about where a particle is found in measurement, not where it is in space.

1.3 Relativity

In Einstein’s 1905 treatment of special relativity the speed of light is necessarily constant because coordinates remote from the origin depend for their definition on the physical behaviour of light. Before one can study physics, one first has to set up a coordinate system. This is done relative to matter, i.e. relative to a reference frame. The “frame” is here defined as the matter used to define coordinates, and includes at least a clock and physical means of determining distance and direction within a neighbourhood. Since coordinates are defined locally with reference to the speed of light, it is necessarily true that light-speed is locally constant (more strictly coordinates are defined with reference to the maximal speed of information transfer – in principle, special relativity would still hold in a universe in which the photon has a very small non-zero mass. In this paper I will not distinguish the speed of light from the maximal speed of information).

As observed by Bondi, 1964, in a modern view of physics, the transmission of virtual photons gives rise to electrodynamics and to all the structures of matter in our immediate environment, including the structure of physical rulers. Thus the constancy of the speed of light can reasonably be interpreted as a *consequence* of the relationist principle that spacetime coordinates only have meaning in so far as they are determined physically by the propagation of light. This interpretation is as described by Einstein, and elucidated by Bondi (1964, 1967). It is not reflected in treatments which simply describe the mathematical properties the Lorentz transform and Minkowski spacetime — such treatments are *not* adequate background for an understanding of relational quantum gravity.

General relativity is sometimes understood as describing a substantive spacetime analogous to Newton’s absolute space, but with geometrical properties of a curved Lorentzian manifold. This view is rejected in relational quantum gravity. Special relativity is imported into general relativity as a local theory. As in Einstein’s 1905 paper, coordinates are defined from the operational definitions of measurement, not as a part of the description of a prior substantive manifold. The definition of a Lorentzian manifold is essentially that the operational definition of coordinates can be applied everywhere within a neighbourhood. The operational definition of coordinates remains fundamental

for the reason that we only need to define vectors in one basis, and they are automatically defined in any basis, and because it also determines the physical metric. Coordinate transformation leads to the general principle of relativity, *local laws of physics are the same irrespective of the coordinate system which a particular observer uses to quantify them*, creating the impression that there are no preferred coordinate systems. However, coordinate transformation is just mathematical manipulation, which adds nothing to physics. At a fundamental level, coordinates are defined within a neighbourhood, according to the operational principles described in special relativity, and the metric is defined from those same operational principles.

General relativity is thus understood as a theory of relationships between empirical results, not as the description of a substantive manifold generalising Newtonian absolute space. Since there is no underlying prior manifold, fields have no fundamental role. The treatment of qed given in RQG II is fundamentally a theory of physical particles. Fields will be understood as mathematical structures without direct physical analogue.

1.4 Quantum Logic

The central problem with relationism has been the difficulty in expressing it formally as axioms for use in mathematical argument. Whereas Newton was able to describe mechanics in three laws, the mathematical implications of relationism were, and have remained, obscure. Here Hilbert space is seen as a formal language which allows us to mathematically describe the behaviour of matter in a universe in which position exists only as a relative quantity ('behaviour' is intended to indicate change with respect to time, and should be understood without spacial connotations). Quantum logic (see e.g. Rescher, 1969) was introduced by Garrett Birkhoff and John Von Neumann (1936) and is sometimes described as applying counter-intuitive truth values to simple propositions. This paper will interpret states as formal conditional clauses, rather than as propositions. The dual space consists of corresponding consequent clauses. The inner product combines clauses to generate formal propositions in the subjunctive mood, showing that the language is a consistent and intuitive extension of two-valued logic and classical probability theory and a natural formalisation of statements about measurements in the subjunctive mood. The principle of superposition is simply logical disjunction in formal language; there is no suggestion of an ontological quantity of magnitude $|\langle x|f \rangle|$ associated with a particular particle.

1.5 Discreteness

Since Newton, the continuum has been induced from the empirical accuracy of physical laws that use it for their expression. But, as Hume argued and Leibniz demonstrated, induction does not provide rigorous scientific proof because an indefinite number of laws can always be found to fit any finite body of data. In this paper the apparatus is not treated from a classical perspective, as in standard Copenhagen. We merely require that the result of measurement of position at given time is always three numbers, and use those numbers to label a condition found in matter. We assume measurement to a level of accuracy limited only by physical law and the ingenuity of the makers of the appara-

tus. In practice measurement results can always be expressed as terminating decimals, and we choose some bounding range and resolution at which to define a basis for a finite dimensional Hilbert space. We can, in principle, use resolutions greater than that of our current apparatus, but observation never permits us to say “for all resolutions” but only “for resolutions up to the current limit of experimental accuracy” (future technology may provide greater resolution, but in any future technology the resolution will still be finite).

It is well understood that a discrete model cannot be manifestly covariant. Manifest covariance will not be applied since it is by definition the case that the apparatus is stationary with respect to the reference frame and affects the measurement result. Since the reference frame is defined by the apparatus, it is meaningless to talk of rotations of the frame unless one is also rotating the apparatus. But in that case one is not rotating vector quantities, but rather redefining them in a new frame. Quantum covariance will also take into account that part of this effect is that the apparatus has a finite resolution, and will restore the principle that local laws of physics are the same in all reference frames.

There are technical advantages in using finite dimensional Hilbert space in that stronger theorems are available and the order of taking limits can be tracked. In certain instances (loop integrals) the order of taking limits is critical as to whether the limit exists. According to a strict reading of the mathematical definition of a limit, what we really mean by a continuum is not necessarily a substantive continuum; it is merely indistinguishable from a continuum at the limit of experimental accuracy. In empirical science we cannot in general say that the result, $f(a)$, of a measurement at $a \in \mathbb{R}^n$ is L . At best we can say that $f(a) \approx L$ to within a margin of error, $\epsilon > 0$; i.e. if the parameters $x \in \mathbb{R}^n$ are close enough to a then we will have $|f(x) - L| < \epsilon$. This is precisely described in the formal mathematical statement of the limit: $\forall \epsilon > 0, \exists \delta > 0$ such that $|x - a| < \delta \Rightarrow |f(x) - L| < \epsilon$. A strict interpretation of empiricism also requires that ϵ cannot be chosen less than permitted by experimental accuracy.

It will be shown that discrete position functions for all coordinate systems are uniquely embedded in smooth wave functions. The continuum equations remove any dependency on a specific measurement apparatus and resolution because they contain embedded within them the solutions for all discrete coordinate systems possible in principal or in practice. Thus, in spite of discreteness, the theory is invariant under changes of lattice, including rotations and improvements to experimental technique.

2 Measurement

2.1 Reference Matter

When a human observer seeks to quantify nature, he chooses some particular matter from which to define a reference frame or chooses certain matter from which he builds his experimental apparatus. He then observes a defined relationship between this specially, but arbitrarily, chosen reference matter and whatever matter is the subject of study. Here measurement is distinguished from a simple count of a number of objects, and is defined to mean a count of units of a measured quantity, where the definition of the unit of measurement invokes comparison between some aspect of the subject of measurement and a property of the reference matter used to define the unit of measurement. The

division between reference matter and subject matter is present in all measurement and appears as the distinction between particle and apparatus in quantum mechanics, and in the definition of position relative to a reference frame in special relativity.

Reference matter is to a large degree arbitrary, and is itself subject to measurement with respect to other matter. D’Inverno (1992) defines a reference frame as a clock, a ruler, and coordinate axes, whereas Rindler (1966) describes a reference frame as a “conventional standard” and discusses the attachment of a frame to definite matter, such as the Earth or the “fixed” stars, while Misner, Thorne and Wheeler (1973) define proper reference frame as a Minkowski coordinate system with a given clock at the origin. Whatever reference matter is used it includes some form of clock, axes, and some means of determining distance, such as a ruler or radar, and it may include any form of apparatus used for physical measurement. In all cases a property is measured relative to other, arbitrarily chosen matter, and the measurement determines a relationship between subject and reference matter, rather than an absolute property of the subject of measurement.

Inertial reference matter is assumed, where inertial is taken to mean that the effect on motion of contact interactions with other matter is negligible. Alternatively inertial coordinates may be calculated from the reference matter. (e.g. a satellite spinning on its axis may be used to determine an inertial reference frame, although it is not itself inertial). This introduces complications in the description, but not complications of a fundamental nature.

2.2 Coordinates

We are particularly interested in measurement of time and position. This is sufficient for the study of many (it has been said all) other physical quantities and we restrict our treatment to those physical quantities that can be reduced to a set of measurements of position, including measurements of position of particles other than the one under study, such as the position of a pointer. For example, a classical measurement of velocity may be reduced to a time trial over a measured distance, and a typical measurement of momentum of a particle involves plotting its path in a bubble chamber, being a set of positions over a time interval. Local distance measurements may be defined by the radar method. Any method of measuring coordinates may be used, calibrated to the radar method, so it is natural to use synchronous spherical coordinates with time as a parameter as in non-relativistic quantum mechanics. For convenience, Cartesian coordinates will be chosen. This simplifies certain formulae, but makes no fundamental difference to the treatment. Any apparatus has a finite resolution and the values written down are triplets of terminating decimals, which can be scaled to integers in units of some bounding resolution. Measured positions are always discrete values, determined by the range and resolution of a measurement apparatus. In practice it is simpler to use an equally spaced lattice, containing very large number N positions given by decimals terminating at some value beyond the best available resolution of any existing apparatus. Margins of error and measurements at lower resolution can be represented using finite sets of such integers. In practice there is also a bound on magnitude. Without loss of generality the same bound, $v \in \mathbb{N}$, is used for each coordinate. Knowledge of the state at any time is

thus restricted to this set of triplets and the results of measurement of position are in a (subset of a) finite region, $D \subset (\chi\mathbb{Z})^3$.

Postulate: The **discrete space coordinate system** is $D \equiv (-\chi\nu, \chi\nu]^3 \subset (\chi\mathbb{Z})^3$ for some $\nu \in \mathbb{N}$, and for some lattice spacing $\chi \in \mathbb{Q}$ with $\chi > 0$.

Let $T \subset \chi\mathbb{Z}$ be a finite discrete time interval such that any particle under study will be measured in D for times $t \in T$.

Postulate: The **discrete spacetime coordinate system** is $S \equiv T \otimes D$ and is calibrated such that the speed of light is 1 radially to the origin.

The coordinate system is a lattice determined by practical considerations. Not every element of D need correspond to a possible measurement result, but D contains as elements or subsets the possible measurement results for a measurement of position with the chosen apparatus. There is no significance in the bound, ν , of a given coordinate system. It is not intended to take either the limit $\nu \rightarrow \infty$ or $\chi \rightarrow 0$, but $\chi\nu$ is large enough to neglect the possibility of particles leaving S . In practice this is always the case since data is discarded from any trial in which there is not both a well defined initial and final state; the probability amplitudes defined below relate to conditional probabilities such that both initial and final states are unambiguously determined (hence there is no detection loophole in Bell tests – in the absence of unambiguous detection this model does not apply).

2.3 Particles

It is sometimes assumed that a particle is localised in space, even if at unknown location. This is not the case here, since a value for the position observable is not assumed to exist between measurements.

Postulate: A **particle** is any physical entity whose position can be measured at given time such that the result of such measurement is a value, $x \in D$, or a neighbourhood $\{x \in D\}$ of negligible size.

Postulate: An **elementary particle** is one which cannot, even in principle, be subdivided into particles for which separate positions can be measured.

It is not necessary to assume the existence of an elementary particle on metaphysical grounds. If there is such a thing as an elementary particle, then its theoretical properties may be determined, and if something in nature exhibits precisely those properties, then we will claim that it is an elementary particle. Quarks may be considered as elementary particles having separate positions in principle, but bound in practice.

2.4 Many Valued Logic

Classical logic applies to sets of statements about the real world which are definitely true or definitely false. For example, when we make a statement,

$\mathcal{A}(x) = \text{The position of a particle is } x,$

we tend to assume that it is definitely true or definitely false. Such statements are said to be **sharp** or **crisp**, meaning that they have truth values from the set $\{0, 1\}$. If it is the case that $\mathcal{A}(x)$ is definitely either true or false then classical logic and classical mechanics apply. Similarly, probability theory gives **Bayesian truth values** from the continuous interval $(1, 0)$ to sentences in the future tense:

$\mathcal{Q}(x) =$ *When a measurement of position is done the result will be x .*

In quantum mechanics we deal with situations in which there has been no measurement and there is not going to be one. $\mathcal{A}(x)$ and $\mathcal{Q}(x)$ are not then legitimate propositions about physical reality. For example we only get interference from Young's slits when there is no way to determine which slit the particle came through. In the absence of measurement we can consider propositions describing hypothetical measurement results, such as the set of propositions of the form:

$\mathcal{P}(x) =$ *If a measurement of position were done the result would be x .*

$\mathcal{P}(x)$ is intuitively sensible, even when no measurement is done, but cannot sensibly be given a crisp truth value. It's truth is distinguished from that of $\mathcal{Q}(x)$ because, when no measurements are to be done, we cannot sensibly discuss the potential frequency of individual measurement results.

2.5 Formal Language

In quantum theory we are not always going to do a measurement, but we want to talk about what would happen if we were to do a measurement, i.e. we need to be able to make statements about hypothetical measurement results. Hilbert space provides a way of discussing levels of truth for statements about hypothetical measurement, like $\mathcal{A}(x)$, in the subjunctive mood. Statements in the subjunctive consist of two clauses, the conditional clause "*If a measurement of position were done, ...*", and the consequent clause "*..., then the result would be x* ". The conditional clause will contain whatever information is known before measurement. This information comes from a prior measurement. We therefore discuss two measurements, the first to determine the condition and the second to determine the outcome, or consequence. We represent the results of these measurements symbolically. The conditional clause, referring to the first measurement, is represented by a ket. It is described as a **formal conditional clause** to indicate that only clauses formally described in the rules are allowed in formal language. **Basic conditional clauses**, on which the language is built, refer to measurements of position:

RULE I. For $x \in D$, $|x\rangle$ is the formal conditional clause "*If measured position at time t were x , ...*".

An actual position found by a real apparatus is described by a set of points in the lattice. To describe this we need to extend the language, by introducing an operator corresponding to OR, represented by the symbol $+$. To express the idea that one possibility is more likely than the other we introduce a weighting. Thus, if the magnitude of a is greater than that of b , then $a|g\rangle + b|f\rangle$ will mean "if measured position were either x or y , but more likely x , ...". We also want to be able to express many possibilities, "*If the particle were found at x or y or z or ...*". This is done recursively in rule II:

RULE II. If $|g\rangle$ and $|f\rangle$ are formal conditional clauses, and a and b are complex numbers, then $a|g\rangle + b|f\rangle$ is a formal conditional clause.

The set of formal conditional clauses, or kets, now has the mathematical structure of an N -dimensional vector space, $\mathbb{H}^1(t)$. $\mathbb{H}^1(t)$ consists of a family of formal conditional clauses concerning the measurement of position of a single particle at time t . Basic conditional clauses, $|x\rangle$, are a basis for $\mathbb{H}^1(t)$. Kets are not strictly states of a particle, but formal conditional clauses describing hypothetical measurement results. They will be referred to as “states”, in keeping with common practice when no confusion arises. The use of a vector space over the complex numbers introduces a degree of freedom which will be used in the description of the evolution of states.

To complete a formal sentence we need to put it together with a consequent clause. Consequent clauses refer to a second measurement, at the same time as the first measurement. To make statements about real measurement results we will also need to know how kets evolve in time, but in the first instance the discussion is restricted to hypothetical measurements at time t . There is no fundamental difference between one measurement and another, so the grammatical structure, *weighted disjunction* described in rule II, applies equally well to consequent clauses. These also form an N -dimensional vector space with $N = 8v^3$, defined from a basis of consequent clauses in one-one correspondence with the basic conditional clauses, or kets, described by rule I. Consequent clauses are represented symbolically by bras:

RULE III. $\langle x|$ is the formal consequent clause “..., then, in a second measurement at the time t , measured position would be x ”.

We put the two clauses together, to make a bracket, representing a statement about measurement at a given time:

RULE IV. $\langle x|y\rangle$ is the statement “If measured position at time t were y , then, in a second measurement at time t , measured position would be x ”.

From observation we know that, if, at some particular time, a particle is measured at position x , then its position is definitely x and it cannot be measured separately at some other position y at the same time. The statement is strictly true or false, depending on whether or not $x = y$.

Postulate: The **truth value** of $\langle x|y\rangle$ is given by a Kronecker delta. We write

$$\langle x|y\rangle = \delta_{xy} \quad (2.5.1)$$

Together with linearity and complex conjugation, (2.5.1) defines an inner product between any two kets, $|f\rangle, |g\rangle \in \mathbb{H}^1(t)$. Thus, $\mathbb{H}^1(t)$ is a Hilbert space, the basic conditional clauses of rule I are an orthonormal basis, and the space of bras is the dual space.

Definition: The **position function** of the ket $|f\rangle \in \mathbb{H}^1$ is the mapping, $D \rightarrow \mathbb{C}$,

$$\forall x \in D, x \rightarrow \langle x|f\rangle. \quad (2.5.2)$$

Later the position function will be identified with the restriction of the wave function to D . It is here termed “position function” because it is discrete and because a wave equation is not assumed.

In this formal language, relative magnitudes are important in weighted logical OR, but absolute magnitude has no meaning. It is easy in common language to construct phrases containing redundant words. “The black piece of coal” is not the same phrase as “the piece of coal”, but both have the same meaning. Similarly, for any complex number a , the clause $|f\rangle$ means exactly the same thing as $a|f\rangle$. When not part of a larger construction containing $+$, a has the role of a redundant word.

The resolution of unity is found by expanding a ket in a normalised basis

$$|f\rangle = \sum_{x \in D} |x\rangle \langle x|f\rangle \quad (2.5.3)$$

Hence

$$1 = \sum_{x \in D} |x\rangle \langle x| \quad (2.5.4)$$

The inner product is strictly a finite sum with N terms, where $N = (2v\chi)^3$ is large. The formal limit $N \rightarrow \infty$ is only to be taken at the final stage of calculation. With this in mind, it is convenient to normalize basis states,

$$\forall x, y \in D, \langle x|y\rangle = \chi^{-3} \delta_{xy} \quad (2.5.5)$$

With this normalisation, the resolution of unity takes the form:

$$1 = \chi^3 \sum_{x \in D} |x\rangle \langle x|. \quad (2.5.6)$$

2.6 Multiparticle States

RULE Va. $| \rangle$ is the formal conditional clause, “*If the first measurement at time t were to find no particle, ...*”.

RULE Vb. $\langle |$ is the formal consequential clause, “*..., then a second measurement at time t would find no particle*”.

Definition: Let \mathbb{H}^0 be the space spanned by $| \rangle$.

\mathbb{H}^0 is a one dimensional vector space. Because multiplication by scalars only has meaning in association with the weighting in OR, there is no difference in meaning between member clauses, $a| \rangle$, of \mathbb{H}^0

Postulate: The space of kets for n particles of the same type is given by the tensor product $\mathbb{H}^n \equiv \bigotimes_n \mathbb{H}^1$

RULE VIa. $|x_1\rangle|x_2\rangle\dots|x_n\rangle$ is the formal conditional clause, “*If, for each of n particles, measured position at time t of the i^{th} particle were x_i , ...*”.

RULE Vb. $\langle x_1|\langle x_2|\dots\langle x_n|$ is the formal consequential clause, “*..., then, for each of n particles in a second measurement at time t , the measured position of the i^{th} particle would be x_i* ”.

Postulate: The space of any number of particles of the same type, γ , is $\mathbb{H}_\gamma \equiv \bigoplus_n \mathbb{H}^n$

The direct sum allows statements about an uncertain number of particles, using weighted logical OR, “*If, for each of n or m particles, but more likely n than m , ...*”, etc. Since an

n particle state cannot be an m particle state, the bracket between states of different numbers of particles is zero. For $|f\rangle = (|f_1\rangle, \dots, |f_n\rangle) \in \mathbb{H}^n$, $|g\rangle = (|g_1\rangle, \dots, |g_n\rangle) \in \mathbb{H}^n$,

$$\langle f|g\rangle = \prod_{i=1}^n \langle f_i|g_i\rangle, \quad (2.6.1)$$

as is required for independent particles by the probability interpretation (section 4.1).

Postulate: The space of particles is $\mathbb{H} \equiv \bigoplus \mathbb{H}_\gamma$.

2.7 Identical particles

If particles are identical, there is no way of telling which is which in a measurement.

RULE VIIa. $|x_1; x_2; \dots; x_n\rangle$ is the formal conditional clause “If, for n identical particles, measured positions were x_1, x_2, \dots, x_n ”.

RULE VIIb. $\langle x_1; x_2; \dots; x_n|$ is the formal consequential clause “then, for n identical particles, measured positions would be x_1, x_2, \dots, x_n ”.

Switching identical particles makes no difference to the physical situation. Multiparticle space is therefore restricted to symmetrical or asymmetrical states.

Definition: Fock space is $\mathbb{F} \equiv \bigoplus_n S\mathbb{H}^n$,

where S means that groups of tensor indices referring to the same type of particle are symmetrised for Bosons and antisymmetrised for Fermions.

3 Momentum Space

3.1 Formal definition

Definition: For a 3-vector, p , at the origin, define the **momentum ket**, $|p\rangle$, as a sum of position states:

$$|p\rangle = \left(\frac{1}{2\pi}\right)^{3/2} \chi^3 \sum_{x \in \mathbb{D}} e^{ix \cdot p} |x\rangle, \quad (3.1.1)$$

where the dot product uses the Euclidean metric. The use of Euclidean metric in (3.1.1) has no direct bearing on a physical metric, and merely defines momentum kets as linear combinations of basic conditional clauses. The inner product with $|x\rangle$ defines a plane wave,

$$\langle x|p\rangle = \left(\frac{1}{2\pi}\right)^{3/2} e^{ix \cdot p}. \quad (3.1.2)$$

Definition: $|p\rangle$ is a **plane wave state** with **momentum** p .

This is the fundamental definition of 3-momentum in this approach. The justification for the definition is that it will be found that p is a conserved quantity which corresponds precisely to the classical notion of momentum.

Definition: Continuum momentum space is the 3-torus,

$$\mathbb{M} \equiv \frac{\pi}{\chi\nu} [-\nu, \nu]^3 \subset \mathbb{R}^3. \quad (3.1.3)$$

There are momentum states $|p\rangle$ in \mathbb{H}^1 for continuum values of $p \in \mathbb{M}$ (since they're just linear combinations of basis kets $|x\rangle$), but a discrete subset of momentum states,

$$\{|p\rangle, p \in \mathbb{M}_D = \mathbb{M} \cap (\chi_p \mathbb{Z})^3\}, \quad (3.1.4)$$

is a basis for \mathbb{H}^1 , where lattice spacing for \mathbb{M}_D is given by $\chi_p = \pi/(\chi\nu)$. Using discrete transforms, Fourier inversion is exact. The resolution of unity in momentum space is

$$\chi_p^3 \sum_{p \in \mathbb{M}_D} |p\rangle\langle p| = 1. \quad (3.1.5)$$

Definition: For the ket, $|f\rangle \in \mathbb{H}^1(t)$, determined by measurement at time $x^0 = t$ using discrete coordinates, \mathbb{D} , the **momentum space wave function** $F : \mathbb{M} \rightarrow \mathbb{C}$ is

$$p \rightarrow F(p) = \langle p|f\rangle. \quad (3.1.6)$$

In particular, for the position ket $|z\rangle$, the momentum space wave function is, for $p \in \mathbb{M}$,

$$p \rightarrow \langle p|z\rangle = \left(\frac{1}{2\pi}\right)^{3/2} e^{-iz \cdot p}. \quad (3.1.7)$$

It is straightforward to show that, for $x, y \in \mathbb{D}$,

$$\int_{\mathbb{M}} d^3p \langle x|p\rangle\langle p|y\rangle = \left(\frac{1}{2\pi}\right)^3 \int_{\mathbb{M}} d^3p e^{-iy \cdot p} e^{ix \cdot p} = \chi^{-3} \delta_{xy} = \langle x|y\rangle. \quad (3.1.8)$$

Thus, Fourier inversion holds using the integral on momentum space; for any $|f\rangle \in \mathbb{H}^1$,

$$\int_{\mathbb{M}} d^3p \langle x|p\rangle\langle p|f\rangle = \int_{\mathbb{M}} d^3p \chi^3 \sum_{y \in \mathbb{D}} \langle x|p\rangle\langle p|y\rangle\langle y|f\rangle = \langle x|f\rangle. \quad (3.1.9)$$

So, we can identify the sum over discrete momenta with an integral over \mathbb{M} ,

$$1 \equiv \chi_p^3 \sum_{p \in \mathbb{M}_D} |p\rangle\langle p| \equiv \int_{\mathbb{M}} d^3p |p\rangle\langle p| \quad (3.1.10)$$

Then for any $|f\rangle \in \mathbb{H}^1$, $q \in \mathbb{M}$

$$\langle q|f\rangle \equiv \chi_p^3 \sum_{p \in \mathbb{M}_D} \langle q|p\rangle\langle p|f\rangle \equiv \int_{\mathbb{M}} d^3p \langle q|p\rangle\langle p|f\rangle. \quad (3.1.11)$$

Thus, for any $p, q \in \mathbb{M}$

$$\langle q|p\rangle = \delta(p - q), \quad (3.1.12)$$

It is perhaps unexpected that the Dirac delta function on the test space of momentum space wave functions has an exact representation as a smooth function,

$$\delta(p - q) \equiv \frac{1}{2\pi} \chi^3 \sum_{x \in \mathbb{D}} e^{ix \cdot (p - q)}. \quad (3.1.13)$$

3.2 Smooth representation

Definition: \mathbb{D} is embedded into the **continuum coordinate system**, \mathbb{C} ,

$$\mathbb{D} \subset \mathbb{C} \equiv [-\chi\nu, \chi\nu]^3 \subset \mathbb{R}^3. \quad (3.2.1)$$

Definition: For any $x \in \mathbb{C}$ we may define the **position ket**

$$|x\rangle = \chi_p^3 \sum_{p \in M_D} |p\rangle \langle p|x\rangle = \int_M d^3p |p\rangle \langle p|x\rangle \quad (3.2.2)$$

Definition: The **wave function** for $|f(t)\rangle \in \mathbb{H}^1(t)$ is the differentiable function $f(t): \mathbb{C} \rightarrow \mathbb{C}$.

$$x \rightarrow f(t, x) = \langle x|f(t)\rangle = \chi^3 \sum_{z \in D} \langle x|z\rangle \langle z|f(t)\rangle \quad (3.2.3)$$

The wave function for $|z\rangle, z \in \mathbb{C}$, is, for $x \in \mathbb{C}$,

$$x \rightarrow f_z(x) = \int_M d^3p \langle x|p\rangle \langle p|z\rangle = \left(\frac{1}{2\pi}\right)^3 \int_M d^3p e^{ix \cdot p} e^{-iz \cdot p} \quad (3.2.4)$$

It is easily verified that for $x, z \in D$

$$f_z(x) = \chi^{-3} \delta_{xz} = \langle x|z\rangle \quad (3.2.5)$$

So, the position function is the restriction of the wave function to D , and, for $z \in D$, there is a one-one correspondence between the wave functions, $f_z(x)$, and basis kets, $|z\rangle$, such that smooth wave functions are a representation of a finite dimensional Hilbert space. For $p, q \in M_D$

$$\int_{\mathbb{C}} d^3x \langle p|x\rangle \langle x|q\rangle = \left(\frac{1}{2\pi}\right)^3 \int_{\mathbb{C}} d^3x e^{-ix \cdot (p-q)} = \chi_p^{-3} \delta_{pq} = \langle p|q\rangle \quad (3.2.6)$$

So, by linearity, we can identify the sum over discrete coordinates with an integral. The identity operator $1: \mathbb{H}^1 \rightarrow \mathbb{H}^1$ can be written

$$1 \equiv \chi^3 \sum_{x \in D} |x\rangle \langle x| \equiv \int_{\mathbb{C}} d^3x |x\rangle \langle x| \quad (3.2.7)$$

Then for any $|f\rangle \in \mathbb{H}^1, y \in \mathbb{C}$

$$\langle y|f\rangle = \chi^3 \sum_{x \in D} \langle y|x\rangle \langle x|f\rangle = \int_{\mathbb{C}} d^3x \langle y|x\rangle \langle x|f\rangle. \quad (3.2.8)$$

and for any $x, y \in \mathbb{C}$

$$\langle y|x\rangle = \delta(x-y) \quad (3.2.9)$$

where the Dirac delta is a smooth function:

$$\delta(x-y) \equiv \chi^3 \sum_{p \in M_D} e^{i(x-y) \cdot p} \equiv \int_M d^3p e^{i(x-y) \cdot p}. \quad (3.2.10)$$

3.3 Bounds

Since coordinate space is discrete, momentum space is the 3-torus M , which is not covariant. The theory would break down if physical momentum could exceed $p_{\max} = \pi/\chi$, where χ is the lower bound of small lattice spacing, not the spacing appropriate to a given apparatus. In conventional units the components of momentum have a theoretical bound $p_{\max} = \pi \hbar c / \chi$. It will be shown in RQG III that the curvature expressed in Einstein's field equation is equivalent to the existence a fundamental discrete unit of proper time, χ , between particle interactions of magnitude twice the Schwarzschild radius, $2GM$, for an elementary particle. For an electron of mass m , $\chi = 4Gm / c^3 = 9.03 \times 10^{-66} \text{s} = 2.71 \times 10^{-57} \text{m}$, where G is the gravitation constant, giv-

ing a theoretical bound on momentum of 2.29×10^{50} eV, or 4.08×10^{14} kg for the energy of a single electron, well beyond any reasonable energy level. Thus, in practice, physical momentum does not approach the bound and there is not an issue.

In fact, there is a much lower bound on energy-momentum since an interaction between a sufficiently high energy electron and any electromagnetic field leads to pair creation (the Greisen–Zatsepin–Kuzmin limit on the energy of cosmic rays is 5×10^{19} eV). It follows from conservation of energy that the total energy of a system is bounded provided that energy has been bounded at some time in the past. This is true whenever an energy value is known since a measurement of energy creates an eigenstate with a definite value of energy. Then momentum is also bounded, by the mass shell condition. The probability of finding a momentum above the bound is zero, and we assume that, for physically realizable states, $\langle p|f \rangle$ vanishes above the bound on each component of momentum. The bound depends on the system under consideration, but without needing to specify a least bound, we may reasonably assume that momentum is always much less than $\pi / (4\chi)$.

A theoretical bound on momentum might introduce a problem of principle for Lorentz transformation. If a high energy electron were boosted beyond the bound it might appear after the boost with a low energy, or with opposite direction of momentum. However, realistic Lorentz transformation means that macroscopic matter (i.e. the reference frame) is physically boosted by the amount of the transformation. For example, for a cubic lattice with spacing equal to the Schwarzschild radius of an electron, a boost in the order of $\pi / (4\chi)$ would require an energy of 2×10^{14} solar masses per kilogram of matter to be boosted. It is therefore reasonable to assume that in any reference frame determined by physical matter there is no other matter with sufficient energy to define a reference frame boosted from the first by more than $\pi / (4\chi)$, so that momentum remains bounded by $\pi / (2\chi)$ in all physically meaningful frames. Thus, in practice, Lorentz transformation cannot boost momentum beyond the level for which it is consistently defined.

The non-physical periodic property of $\langle p|f \rangle$ can be removed by the substitution $\Theta_M(p)\langle p|f \rangle \rightarrow \langle p|f \rangle$, where $\Theta_M(p) = 1$ if $p \in M$ and $\Theta_M(p) = 0$ otherwise. The wave function (3.1.9) may then be replaced with the standard form in relativistic quantum mechanics:

$$f(x) = \left(\frac{1}{2\pi}\right)^{3/2} \int_{\mathbb{R}^3} d^3p \langle p|f \rangle e^{-ix \cdot p} \quad (3.3.1)$$

4 Observable Quantities

4.1 Probability Interpretation

To make the formal language precise, we must assign numerical values to the complex numbers introduced in rule II, i.e. we must determine magnitude and phase. Phase contains information on the evolution of kets, and will be considered later. Magnitude will be determined from probability. It only makes sense to talk about probability when we are actually going to do a measurement. When we are actually going to do the measurement, a statement about hypothetical measurement, in the subjunctive mood, automatically becomes a statement about real measurement, in the future tense. This

being the case, truth values for hypothetical results must be replaced by truth values for future events, i.e. probabilities, when experiments are actually done.

In a typical measurement in quantum mechanics we study a particle in near isolation. The suggestion is that there are too few ontological relationships to create the property of position and that measurement introduces interactions which generate position. In this case, prior to measurement, position does not exist and the state of the system is not labelled by a position ket. Instead, Hilbert space is used to provide a label containing information about the about the probability of what would happen in measurement. To associate a ket, $|f\rangle$, with a particular physical state it is necessary and sufficient to specify the magnitude and phase of $\langle x|f\rangle$ from empirical data. In a scientific measurement of position we set up many repetitions of a system described by the initial measurement results, f , and record the frequency of each result, x . For a large number of repetitions the relative frequency of x tends to the probability, $P(x|f)$, of finding the particle at x . The amplitudes of the components $\langle x|f\rangle$ are determined from the probabilities of measurement results (not the other way about):

Postulate: For the state $|f\rangle \in \mathbb{H}^1$, the magnitudes of the coefficients, $\langle x|f\rangle$ are defined such that

$$\frac{|\langle x|f\rangle|^2}{\langle f|f\rangle} = P(x|f). \quad (4.1.1)$$

Definition: If $\langle f|f\rangle = 1$ then $|f\rangle$ is said to be **normalised**.

4.2 Measurement

Since only a general principle has been used that it is possible to measure position, it is necessary to discuss other observables. The question as to what other observables exist cannot be discussed until after the treatment of interactions between particles (RQG II). It will then be assumed that all observables are a product of physical laws arising from particle interactions. A full analysis of a given measurement would require that the measurement apparatus as well as the system being measured be treated as a multiparticle system in Fock space, in which time evolution for the interacting theory is known. Here general considerations are discussed on the assumption that interactions will be described by linear maps on Fock space and that measurement is always a physical process describable in principle as a combination of interaction operators. For qed this will mean that all observables depend only on the electric current operator and the photon field operator. A complete resolution of the measurement problem would demonstrate the projection postulate for any given apparatus and has not been given. The argument given below makes the projection postulate reasonable by reducing all measurement to measurement of position. The view is that if we find a physical process satisfying the projection postulate then we may say it defines an observable quantity.

Measurement has two effects on the state of a particle, altering it due to the interaction of the apparatus with the particle, and also changing the information we have about the state. New information causes a change of state even in the absence of physical change because the state is just a label for available information. Then the collapse of the wave function is in part the effect of the apparatus on the particle, and in part the effect on con-

ditional probability when the condition becomes known. This inverts the measurement problem; collapse represents a change in information due to a new measurement but Schrödinger's equation requires explanation – interference patterns are real. The requirement for a wave equation will be found in section 5.

Classical probability theory describes situations in which every parameter exists, but some are not known. Probabilistic results come from different values taken by unknown parameters. We have a similar situation here, but now the unknowns are not describable as parameters. We assume no relationships between particles bar those generated by physical interaction. An experiment is described as a large configuration of particles incorporating the measuring apparatus as well as the process being measured. The configuration has been partially determined by setting up the experimental apparatus, reducing the possibilities to those with definite outcomes to the measurement. It is impossible, even in principle, to determine every detail of the configuration since the determination of each detail requires measurement, which in turn requires a larger apparatus containing new unknowns in the configuration of particles. Thus there is always a lack of determination of initial conditions leading to randomness in the outcome, whether or not there is a fundamental indeterminism in nature.

When we do a measurement, K , we get a definite result, a terminating decimal or n -tuple of terminating decimals read off the measurement apparatus. Let the possible results be $k_i \in \mathbb{Q}^n$ for $i = 1 \dots N$. We assume that the dimension of \mathbb{H}^1 is greater than N ; this must be so if all measurements are reducible to measurements of position, and can be ensured by the choice of a lattice finer than the resolution of measurement. Each physical state is associated with a ket, labelled by the measurement result, so that if the measured result is k_i then the state is $|k_i\rangle$. The empirical determination of $|k_i\rangle$ as a member of \mathbb{H}^1 requires that we draw from experimental data the value of the inner product $\langle k_i | f \rangle$ for an arbitrary state, $|f\rangle$. Without loss of generality $|k_i\rangle$ and $|f\rangle$ are normalised. By assumption, measurement of K is reducible to a set of measurements of position, so that each k_i is in one to one correspondence with the positions y_i of one or more particles used for the measurement (e.g. y_i may be the positions of one or more pointers). Then,

$$|\langle k_i | f \rangle|^2 = |\langle y_i | f \rangle|^2 = P(y_i | f) = P(k_i | f). \quad (4.2.1)$$

is the probability that a measurement of K has result k_i , given the initial state $|f\rangle \in \mathbb{H}^1$. It follows from (2.5.1) that

$$\langle k_i | k_j \rangle = \delta_{ij} = \langle y_i | y_j \rangle. \quad (4.2.2)$$

So, if the result is k_i it is definitely k_i and cannot at the same time be k_j with $i \neq j$.

Measurement with result, k_i , implies a physical action on a system and is represented by the action of an operator, K_i , on Hilbert space. If a quantity is measurable we require that there is an element of physical reality associated with its measurement, by which we mean that the configuration of particles necessarily becomes such that the quantity has a well defined value. In practice this means that, in the limit in which the time between two measurements goes to zero, a second measurement of the quantity necessarily gives the same result as the first. It follows that K_i is a projection operator (the projection postulate),

$$K_i = |k_i\rangle\langle k_i|. \quad (4.2.3)$$

The projection postulate is too restrictive to describe all numerical quantities used in the classical description of nature, and will be relaxed after a discussion of expectations.

4.3 Observable Operators

The expectation of the result from a measurement of K , given the initial normalised state, $|f\rangle \in \mathbb{H}^1$, is

$$\langle K \rangle \equiv \sum_i k_i P(k_i|f) = \sum_i \langle f|k_i\rangle k_i \langle k_i|f\rangle = \langle f|K|f\rangle. \quad (4.3.1)$$

Postulate: The Hermitian operator, $K = \sum_i |k_i\rangle k_i \langle k_i|$, is called an **observable**. k_i is the **value** of K in the state $|k_i\rangle$.

Using (4.2.1) the probability that operators describing the interactions comprising the measurement of K combine to give the result K_i is

$$P(k_i|f) = |\langle k_i|f\rangle|^2 = \langle f|k_i\rangle \langle k_i|f\rangle = \langle f|K_i|f\rangle. \quad (4.3.2)$$

Then $P(k_i|f)$ can be understood as a classical probability function, where the random variable runs over the set of projection operators, K_i , corresponding to the outcomes of the measurement. The physical interpretation is that each K_i represents a set of unknown configurations of particle interactions in measurement, namely that set of configurations leading to the result k_i .

4.4 The Canonical Commutation Relation

We may define the momentum operator $P^a = -i\partial^a: \mathbb{H}^1 \rightarrow \mathbb{H}^1$ for $a = 1, 2, 3$,

$$\begin{aligned} P^a: |f\rangle &\rightarrow -\int_{\mathbb{C}} d^3x |x\rangle i\partial^a \langle x|f\rangle = -\int_{\mathbb{C}} d^3x |x\rangle i\partial^a \chi_p^3 \sum_{p \in \mathbb{M}_D} \langle x|p\rangle \langle p|f\rangle \\ &= \chi_p^3 \sum_{p \in \mathbb{M}_D} |p\rangle p^a \langle p|f\rangle. \end{aligned} \quad (4.4.1)$$

Clearly P^a is Hermitian. Note that

$$P^a \neq \int_{\mathbb{M}} dp |p\rangle p^a \langle p|f\rangle. \quad (4.4.2)$$

The position operator, $X^a: \mathbb{H} \rightarrow \mathbb{H}$, is given for $a = 1, 2, 3$ by

$$X^a|f\rangle = \chi^3 \sum_{x \in \mathbb{D}} |x\rangle x^a \langle x|f\rangle. \quad (4.4.3)$$

It is not possible to differentiate under the sum in the product $P^a X^b$. From the property that the trace of a commutator in finite dimensional Hilbert space vanishes,

$$\text{Tr}([X^a, P^b]) = 0, \quad (4.4.4)$$

it is seen that we do not have the canonical commutation relation,

$$[X^a, P^b] \neq i\delta_{ab}. \quad (4.4.5)$$

Depending on the precise chosen definition of a the discrete momentum operator we have

$$[X^a, P^b] = \frac{i}{2} \begin{bmatrix} 0 & 1 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & \dots \\ 0 & 1 & 0 & 1 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}. \quad (4.4.6)$$

This is replaced by a delta function in a continuum treatment (taking a limit is highly non-trivial, but it is neither appropriate nor necessary to give a formal construction of rigged Hilbert space). If we formally define \tilde{X} by

$$\tilde{X}^a |f\rangle = \int_{\mathbb{C}} dx |x\rangle x^a \langle x | f \rangle. \quad (4.4.7)$$

Then,

$$P^b \tilde{X}^a |f\rangle = - \int_{\mathbb{C}} dx |x\rangle i \delta_{ab} \langle x | f \rangle - \int_{\mathbb{C}} dx |x\rangle x^a i \partial^b \langle x | f \rangle = -i \delta_{ab} - \tilde{X}^a P^b |f\rangle. \quad (4.4.8)$$

So,

$$[\tilde{X}^a, P^b] = i \delta_{ab}. \quad (4.4.9)$$

and we conclude that $X^a \neq \tilde{X}^a$ and that $\tilde{X}^a |f\rangle \notin \mathbb{H}^1$.

4.5 Classical Correspondence

In the classical correspondence we study the behaviour of systems containing a large number, N , of quantum motions (this is sometimes called the thermodynamic limit). A classical property is the expectation, (4.3.1), of the corresponding observable in the limit $N \rightarrow \infty$ (not $\hbar \rightarrow 0$ as sometimes stated; Planck's constant is simply a change of scale from natural to conventional units and it would be meaningless to let it go to zero). For example, the centre of gravity of a macroscopic body is a weighted average of the positions of the elementary particles which constitute it. Schrödinger's cat is definitely either alive or dead because, consisting as it does of a large number of elementary particles, its properties are expectations obeying classical laws derived from (4.3.1), but the state simply encodes probability and the cat may be described as a superposition until the box is opened. A precise treatment of the time evolution of classical quantities requires the prior development of an interacting theory which will be the subject of RQG II. It will be shown there that determinate laws obtain for classical quantities. In this paper we will simply assume determinate laws for expectations in the large number limit.

Postulate: A **measurement** of a physical quantity is any physical process such that a determination of the quantity is possible in principle.

In keeping with the considerations of section 4.2, we assume that the existence of a value for an observable quantity depends only on the configuration of matter. If a configuration of matter corresponds to an eigenstate of an observable operator then the value of that observable exists independently of observation and is given by the corresponding eigenvalue. In classical physics there is sufficient information to determine the motion

at each instant between the initial and final state, up to experimental accuracy. Intermediate states are similarly determinate and may be calculated in principal by the processing of data already gathered, or which could be gathered without physically affecting the measurement. So in classical physics intermediate states may be regarded as measured states, and we may say that they are **effectively measured**.

The projection postulate is required if the results of measurement are to be used to name states in Hilbert space, but classical quantities can also be defined from Hermitian operators when this is not the case. To say that a Hermitian operator has a well defined value in a given state, a measurement should necessarily yield that value as the expectation of the operator. This is weaker than the projection postulate, which requires an eigenstate (in which the value is trivially given by the expectation).

Postulate: For states consisting of large numbers of particles, the **classical value** of an observable quantity is given by the expectation of the corresponding Hermitian operator (irrespective of whether the state is an eigenstate).

5 Evolution

The arguments in this section are known. They are reviewed to show that the continuum equations do not require prior space or spacetime, and because it is often not clearly understood that the Schrödinger equation is strictly a theorem, not a postulate. It results *solely* from the requirements of the probability interpretation and has no direct physical content as an equation of motion.

5.1 Linearity of Time Evolution

The inner product allows us to calculate probabilities for the outcome of a measurement provided that we know the ket describing hypothetical measurement at the time of measurement. This is only useful if we can calculate the ket at any time, t , from a known previous measurement result. Hilbert space refers to measurement at time, t , so that $|f(t)\rangle \in \mathbb{H}(t)$, where t is a parameter and we isomorphically identify $\mathbb{H}(t) = \mathbb{H}$ for all t . The position state $|x\rangle$ at time $x^0 = t$ will be denoted by $|t, x\rangle$.

Postulate: If at time t_0 the ket is $|f(t_0)\rangle$, then the ket at time t is given by the **time evolution operator**, $U(t, t_0): \mathbb{H} \rightarrow \mathbb{H}$, such that

$$|f(t)\rangle = U(t, t_0)|f(t_0)\rangle. \quad (5.1.1)$$

If the state at time t_0 was either $|f(t_0)\rangle$ or $|g(t_0)\rangle$, then it will evolve into either $|f(t)\rangle$ or $|g(t)\rangle$ at time t . Any weighting in OR will be preserved, i.e., if

$$|h(t_0)\rangle = a|f(t_0)\rangle + b|g(t_0)\rangle \quad (5.1.2)$$

then

$$|h(t)\rangle = a|f(t)\rangle + b|g(t)\rangle \quad (5.1.3)$$

So, U is linear

$$U(t, t_0)(a|f(t_0)\rangle + b|g(t_0)\rangle) = aU(t, t_0)|f(t_0)\rangle + bU(t, t_0)|g(t_0)\rangle. \quad (5.1.4)$$

5.2 Continuity of Time Evolution

Irrespective of whether a model of discrete particle interactions might appear continuous on the large scale, the evolution of kets is expected to be continuous because kets are not physical states of matter, but are rather probabilistic statements about what might happen in measurement, given current information. Probabilities describe our *ideas* concerning the likelihood of events. Whether or not reality is fundamentally discrete, changes in probability can be properly described on a mathematical continuum. A discrete interaction will not lead to a discrete change in probability because we do not have exact information on when the interaction takes place. This being so, time evolution will be modelled by a continuous operator valued function of time, U , (see section 3.2). Together with the considerations below, continuity is sufficient to ensure differentiability. Since local laws of physics are always the same, and U does not depend on the state on which it acts, the form of the evolution operator for a time interval t , $U(t) = U(t + t_0, t_0)$ does not depend on t_0 . We require that the evolution in an interval $t_1 + t_2$ is the same as the evolution in t_1 followed by the evolution in t_2 , and is also equal to the evolution in t_2 followed by the evolution in t_1 ,

$$U(t_2)U(t_1) = U(t_2 + t_1) = U(t_1)U(t_2) \quad (5.2.1)$$

In a zero time interval, there is no evolution. So, $U(0)$ does not change the state.

$$U(0) = 1 \quad (5.2.2)$$

Using a negative value of t reverses time evolution (put $t = t_1 = -t_2$).

$$U(-t) = U(t)^{-1}. \quad (5.2.3)$$

5.3 The Schrödinger equation

Since states can be chosen to be normalised we may require that U conserves the norm, i.e. for all $|g\rangle$,

$$\langle g|U^\dagger U|g\rangle = |U|g\rangle|^2 = ||g\rangle|^2 = \langle g|g\rangle \quad (5.3.1)$$

This is sufficient to show that U is unitary (appendix A). Thus the conditions of Stone's theorem (appendix B) are satisfied and we have that there exists a Hermitian operator H , known as the Hamiltonian, such that

$$\dot{U}(t) = -iHU(t) \quad (5.3.2)$$

which has solution

$$U(t) = e^{-iHt} \quad (5.3.3)$$

Theorem: The wave function satisfies the Schrödinger equation

$$\partial_0 f(t, x) = -iHf(t, x) \quad (5.3.4)$$

Proof: Differentiate the wave function using Stone's theorem,

$$\begin{aligned} \partial_0 f(t, x) &= \langle x|\dot{U}|f(0)\rangle = \langle x|-iHU(t)|f(0)\rangle \\ &= \langle x|-iH|f(t)\rangle = -iHf(t, x) \end{aligned} \quad (5.3.5)$$

If we impose the mass shell condition, $E^2 = (p^0)^2 = m^2 + \mathbf{p}^2$ for some constant m , and replace 3-vectors with 4-vectors, then a plane wave state (3.1.2) is a solution of the Schrödinger equation with $H = E$. Thus, p does not change in time, establishing New-

ton's first law. E is identified with energy and m with mass. It follows that for an initial state $|f\rangle$ with momentum space wave function $\langle p|f\rangle$, the general solution is

$$f(x) = \left(\frac{1}{2\pi}\right)^{3/2} \int d^3p \langle p|f\rangle e^{-ix \cdot p} \quad (5.3.6)$$

Thus the discrete position function (2.5.2) is uniquely embedded into the smooth wave function, (3.3.1). Solving the Schrödinger equation extends the wave function to \mathbb{R}^4 (5.3.6). Then the position function in any discrete coordinate system is found restricting to discrete values. Thus we do not require the existence of a physical continuum to define quantum theory using smooth wave functions.

It will be observed that Newton's first law holds in coordinate space, C , not in curved spacetime. The implications for the reconciliation of quantum theory with general relativity are the subject of RQG III.

6 Quantum Covariance

6.1 The General Principle of Relativity

If time and position are not properties of prior space or spacetime, but only of relationships found in matter, then it follows that the fundamental properties of elementary particles have no dependency on time or position. This is expressed in the principle that, *the fundamental behaviour of matter is always and everywhere the same*. Incorporated in this law is the notion that local, physically released, coordinate systems may always be established by an observer in the same way. From this we may infer the general principle of relativity, *local laws of physics are the same irrespective of the coordinate system which a particular observer uses to quantify them*. In classical physics, laws which are the same in all coordinate systems are most easily expressed in terms of invariants, known as tensors. Then the most directly applicable form of the principle of general relativity is the principle of general covariance, *the equations of physics have tensorial form*. General covariance applies to classical vector quantities under the assumption that they are unchanged by measurement. But in quantum mechanics measured values arise from the action of the apparatus on the quantum system, creating an eigenstate of the corresponding observable operator and we cannot generally assume the existence of a tensor independent of measurement. In practice a change of reference frame necessitates a change of apparatus (either by accelerating the apparatus or by switching to a different apparatus). A lattice describes possible values taken from measurement by a particular apparatus. Eigenstates of displacement are determined by this lattice, i.e. by the properties and resolution of a particular measuring apparatus. So, in general, eigenstates in one frame are not simultaneously eigenstates of a corresponding observable in another frame using another apparatus (c.f. non-commutative geometry, Connes, 2000). For the same reason classical tensor quantities do not, in general, correspond to tensor observables.

6.2 Quantum Covariance

The broad meaning of **covariance** is that it refers to something which varies with something else, so as to preserve certain mathematical relations. If covariance is not now

to be interpreted as manifest covariance or general covariance as applicable to the components of classical vectors, then a new form of covariance, **quantum covariance**, is required to express the principle of general relativity, that local laws of physics are the same in all reference frames. Quantum covariance will mean that local laws of physics have the same form in any reference frame but not that the same physical process may be described identically in different reference frames, since the reference frame, i.e. the choice of apparatus, can affect both the process under study and the description of that process. Since coordinates are determined by physical measurement which has finite resolution, under transformation of the coordinate system (passive Lorentz transformation) there is also a change of basis for Hilbert space. Quantum covariance observes that, since the choice of basis is arbitrary, and Hilbert space contains a continuum of states $|x\rangle$ for $x \in \mathbb{R}^3$, any breaking of manifest covariance by the choice of basis is irrelevant.

Postulate: Quantum covariance will mean that the wave function, (5.3.6), is defined on a continuum, while the inner product, (2.5.1), is discrete, and that, in a change of reference frame, the lattice and inner product appropriate to one reference frame are replaced with the lattice and inner product of another.

Thus, from an initial position function defined on C , the position function at any time is given by

$$\langle x|f\rangle = f(x)|_S, \quad (6.2.1)$$

and if, in a change of reference frame, the spacetime coordinate system S is replaced by S' , the new position function is given by

$$\langle x|f\rangle = f(x)|_{S'}. \quad (6.2.2)$$

We have seen that the consistency of quantum covariance is ensured if the support of $\langle p|f\rangle$ is bounded as described in (3.3).

The general form of a linear operator, O on \mathbb{H} , is, for some complex valued function $O(x, y)$

$$O = \chi^3 \sum_{x, y \in D} |x\rangle O(x, y) \langle y|. \quad (6.2.3)$$

According to quantum covariance, this expression has an invariant form under a change of reference frame. This will be important for the definition of quantum fields in RQG II, since these are operators and are not manifestly covariant as is usually assumed. The invariance of operators under rotations is perhaps at first a little surprising, particularly when one considers the presumed importance of manifest covariance in axiomatic quantum field theory. It may be clarified a little with a nautical analogy. On a boat the directions fore, aft, port and starboard are invariant because they are defined with respect to the boat. Similarly operators are necessarily defined with respect to chosen reference matter and have an invariant form with respect to reference matter.

7 Conclusions

It has been established that formal conditional clauses about hypothetical measurement results have the natural structure of a finite dimensional Hilbert space in which the inner product can be understood as complex truth values for statements in the subjunc-

tive mood. Coefficients are constrained by probabilities which apply when hypothetical measurements are replaced by actual measurements and the subjunctive mood is replaced by a factual conditional.

It is shown that for any coordinate system the position function can be embedded into a smooth wave function obeying the Schrödinger equation. Wave functions are thus directly related to probabilities and do not describe an objective property of matter. Instantaneous collapse of the wave function is merely the collapse of a conditional probability when the condition becomes known. Thus, Schrödinger's cat is not an objective superposition of quantum states, but simply a probabilistic statement that if the box were to be opened there would be a 50-50 probability of finding the cat alive or dead.

The fact that the underlying Hilbert space is finite dimensional has implications for the construction of qed and for the ultraviolet divergences and the Landau pole. This will be studied in RQG II. The canonical commutation relation does not hold in finite dimensional Hilbert space, but can be said to hold in some approximation. In the limit $\chi \rightarrow 0$, $v \rightarrow \infty$, $\langle x | y \rangle$ is not defined. Consistency requires that physical predictions are independent of χ and v (for sufficiently large χ^{-1} and v). Since physical momentum is bounded by considerations of energy conservation, momentum space wave functions representing physical states have bounded support. Provided that χ is sufficiently small, the bound, π/χ , on momentum has no physical import. A form of covariance can be preserved because discrete coordinates, D , are naturally determined from the measurement apparatus used to define the reference frame. In a change of reference frame, the discrete coordinates and inner product appropriate to one apparatus is naturally replaced with the discrete coordinates and inner product appropriate to another.

A physical metric has not been introduced in this paper. As a result, Newton's first law holds in coordinate space, D , which is defined from physical procedures such that the speed of light is constant. Gravity will be introduced in RQG III. The formulation has observational consequences in astrophysics and cosmology, which are the subject of RQG IV. The treatment of expansion has implications for missing mass, the cosmological constant, lensing, galaxy rotation curves and the anomalous Pioneer blue shift. Good agreement will be reported between predictions and observation.

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Appendices

Appendix A Unitarity of U

Since states can be chosen to be normalised we may require that U conserves the norm, i.e., for all $|g\rangle$,

$$\langle g|U^\dagger U|g\rangle = \langle g|g\rangle.$$

Hence, applying this to $|g\rangle + |f\rangle$,

$$(\langle g| + \langle f|)U^\dagger U(|g\rangle + |f\rangle) = (\langle g| + \langle f|)(|g\rangle + |f\rangle).$$

By linearity of U

$$(\langle g|U^\dagger + \langle f|U^\dagger)(U|g\rangle + U|f\rangle) = (\langle g| + \langle f|)(|g\rangle + |f\rangle).$$

By linearity of the inner product

$$\begin{aligned} \langle g|U^\dagger U|g\rangle + \langle g|U^\dagger U|f\rangle + \langle f|U^\dagger U|g\rangle + \langle f|U^\dagger U|f\rangle \\ = \langle g|g\rangle + \langle g|f\rangle + \langle f|g\rangle + \langle f|f\rangle \end{aligned}$$

$$\langle g|U^\dagger U|f\rangle + \langle f|U^\dagger U|g\rangle = \langle g|f\rangle + \langle f|g\rangle.$$

Similarly

$$(\langle g| - i\langle f|)U^\dagger U(|g\rangle + i|f\rangle) = (\langle g| - i\langle f|)(|g\rangle + i|f\rangle)$$

$$\begin{aligned} \langle g|U^\dagger U|g\rangle + i\langle g|U^\dagger U|f\rangle - i\langle f|U^\dagger U|g\rangle + \langle f|U^\dagger U|f\rangle \\ = \langle g|g\rangle + i\langle g|f\rangle - i\langle f|g\rangle + \langle f|f\rangle \end{aligned}$$

$$\langle g|U^\dagger U|f\rangle - \langle f|U^\dagger U|g\rangle = \langle g|f\rangle - \langle f|g\rangle.$$

Combining these results shows that U is unitary,

$$\langle g|U^\dagger U|f\rangle = \langle g|f\rangle.$$

Appendix B Stone's Theorem

The derivative of U is

$$\begin{aligned}\dot{U} &= \frac{dU}{dt} = \lim_{dt \rightarrow 0} \frac{U(t+dt) - U(t)}{dt} = \lim_{dt \rightarrow 0} \frac{U(dt)U(t) - U(t)}{dt} \\ &= \left(\lim_{dt \rightarrow 0} \frac{U(dt) - 1}{dt} \right) U(t) = U(t) \left(\lim_{dt \rightarrow 0} \frac{U(dt) - 1}{dt} \right).\end{aligned}$$

This prompts the definition of the Hamiltonian operator, which has no formal dependency on t .

Definition: The **Hamiltonian operator** $H: \mathbb{H} \rightarrow \mathbb{H}$ is given by

$$H = i \left(\lim_{dt \rightarrow 0} \frac{U(dt) - 1}{dt} \right).$$

We have

$$\dot{U}(t) = -iHU(t) = -iU(t)H.$$

So

$$-iH = U^\dagger \dot{U} = \dot{U} U^\dagger.$$

Since U is unitary, for a small time dt ,

$$U^\dagger(t+dt)U(t+dt) = 1$$

$$[U^\dagger(t) + \dot{U}^\dagger(t)dt][U(t) + \dot{U}(t)dt] \approx 1.$$

Ignoring terms in squares of dt , and using $-iH = U^\dagger \dot{U}$, $iH = \dot{U} U^\dagger$,

$$U^\dagger(t)U(t) - iH^\dagger dt + iHdt \approx 1.$$

Using unitarity of U , we find that H is Hermitian, $H = H^\dagger$. We have the differential equation

$$\dot{U}(t) = -iHU(t)$$

which has solution,

$$U(t) = e^{-iHt}.$$

This result was first proved by proved by Marshall Stone (1932).