Relational Quantum Gravity II: Quantum Electrodynamics

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ABSTRACT
Context. While causal perturbation theory and lattice regularization allow rigorous treatment of the ultraviolet divergences in qed, they do not resolve the Landau pole, or address questions of physical interpretation. Relational Quantum Gravity I (RQG I) presented an interpretation of quantum mechanics as a theory of measurements of particles, and found a representation of N-dimensional Hilbert space, for finite N, in which vectors are smooth wave functions such that differential operators are defined and a form of covariance is obeyed.
Aims. To construct quantum electrodynamics from quantum mechanics as formulated in RQG I.
Methods. Quantum field operators are defined from creation and annihilation operators on Fock space, obeying quantum covariance and locality, and suitable for a description of particle interactions under the Feynman-Stückelberg interpretation.
Results. The formulation is consistent and yields predictions for large N which are identical to those of quantum electrodynamics with all loop divergences removed. Quantum fields are operator valued functions, not distributions. The Landau pole can be avoided by introducing a physical cut-off in the form of a minimal proper time between discrete particle interactions. Maxwell's equations and the Lorentz force law are derived in the classical correspondence, showing that bare mass and charge are equal to their physical values.

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1 Introduction

1.1 Background

Particle theoretic qed has been largely out of favour for more than half a century (Feynman being a notable exception; Schweber, 1994), but there have been many developments in understanding during that time. Among the problems particle qed has to face are the requirement of a positive definite norm for valid probabilities, the indefinability of the equal point multiplication between field operators, loop divergences, and the Landau Pole. The purpose of the present paper is to review the particle theoretic, or Fock space, formulation of quantum electrodynamics in the context of the formulation of quantum theory given by Francis (2009; RQG I) and to show that a non-perturbative model exists (in the strict mathematical sense), that Maxwell's equations and the Lorentz force law are derived in the classical correspondence, and that bare mass and charge are equal to their physical values. The construction does not alter the predictions of perturbative qed.

Although a finite dimensional basis is used for Hilbert space, states $|x\rangle$ are defined for continuum values $x \in \mathbb{R}^3$, and a form of covariance is obeyed. Using finite dimensional Hilbert space, fields are operator valued functions, not distributions. they describe the potential for the creation or annihilation of a particle in an interaction. This cannot be reconciled to interpretational statements which are sometimes made about models which arise from the quantization of classical fields, such as "*The free field describes particles which do not interact*" (Glimm & Jaffe, 1987, p 100) or "*In its mature form, the idea of quantum field theory is that quantum fields are the basic ingredients of the universe, and particles are just bundles of energy and momentum of the fields*" (Weinberg, 1996).

1.2 Quantum Covariance

RQG I found a representation of a finite dimensional Hilbert space, $\mathbb{H}^1$, using smooth wave functions and containing kets corresponding to the measurement results of a single particle at given time. For some $\nu \in \mathbb{N}$, and for some lattice spacing $\chi \in \mathbb{Q}$ with $\chi > 0$, the **discrete coordinate system**,

$$\mathrm{D} \equiv (-\chi\nu, \chi\nu\,]^3 \subset (\chi\mathbb{Z})^3, \tag{1.2.1}$$

is embedded into the **continuum coordinate system**,

$$\mathrm{C} \equiv [-\chi\nu, \chi\nu\,]^3 \subset \mathbb{R}^3. \tag{1.2.2}$$

For the finite discrete time interval, $\mathrm{T} \subset \chi\mathbb{Z}$, such that any particle under study will be measured in D for times $t \in \mathrm{T}$, the **discrete spacetime coordinate system** is

$$\mathrm{S} \equiv \mathrm{T} \otimes \mathrm{D}$$

and is calibrated such that the speed of light is 1 radially to the origin.

For $\{|x\rangle, x \in \mathrm{D}\}$ is a basis for $\mathbb{H}^1$. For $x \in \mathrm{C}$, $|x\rangle$ is defined as a linear combination of basis states. A discrete subset, $\mathrm{M_D}$, of **continuum momentum space,**

$$\mathrm{M} \equiv \frac{\pi}{\chi\nu}[-\nu, \nu]^3 \subset \mathbb{R}^3 \tag{1.2.3}$$

contains a basis of momentum states,

$$\{|p\rangle, p \in \mathrm{M_D} = \mathrm{M} \cap (\chi_p\mathbb{Z})^3, \chi_p = \pi/(\chi\nu)\}. \tag{1.2.4}$$

The identities, for all $|f\rangle \in \mathbb{H}^1$,

$$|f\rangle \equiv \chi^3 \sum_{x \in \mathrm{D}} |x\rangle\langle x|f\rangle \equiv \int_{\mathrm{C}} d^3x |x\rangle\langle x|f\rangle, \tag{1.2.5}$$

$$|f\rangle \equiv \chi_p^3 \sum_{\boldsymbol{p} \in \mathrm{M_D}} |p\rangle\langle p|f\rangle \equiv \int_{\mathrm{M}} d^3p |p\rangle\langle p|f\rangle. \tag{1.2.6}$$

were established. For continuum positions, $x, y \in \mathrm{C}$, and momenta, $p, q \in \mathrm{M}$ the inner products $\langle y|x\rangle$ and $\langle q|p\rangle$ are smooth functions which act as Dirac delta functions under the integral, and which become Kronecker deltas in the restriction to the position and momentum bases. It was seen that the Schrödinger equation applies and that energy-momentum is given by the operator $P^a = -i\partial^a\colon \mathbb{H}^1 \to \mathbb{H}^1$,

$$P^a\colon |f\rangle \to -\int_{\mathrm{C}} d^3x |x\rangle i\partial^a\langle x|f\rangle = \chi_p^3 \sum_{\boldsymbol{p}\in \mathrm{M_D}} |p\rangle p^a\langle p|f\rangle\,. \tag{1.2.7}$$

The canonical commutation relation appears when the sum is replaced by an integral, but the result of this replacement is not in $\mathbb{H}^1$.

It is observed that, due to energy conservation and pair creation, physical momentum space wave functions have bounded support, where the bound is much less than the bound on M, and that in the probability interpretation, the discrete form of the inner product is directly related to the physical apparatus used to determine the reference frame. **Quantum covariance** was defined to mean that the wave function is defined on a continuum, while the inner product is discrete, and that, in a change of reference frame, the lattice and inner product appropriate to one reference frame are replaced with the lattice and inner product of another. In relational quantum gravity, quantum covariance is taken to be the correct expression of the principle of general relativity in the quantum domain.

1.3 Outline

Notations described in RQG I will be used. Based on the quantum covariant formulation of quantum mechanics the construction proceeds under broadly conventional lines. Section 2, *Particles*, introduces spin and reviews the photon and the Dirac particle. Section 3, *Interactions* introduces the interaction Hamiltonian. The Hamiltonian density is introduced, and locality requirement is seen from the perturbation expansion. Section 4, *Field Theory* defines the Dirac field operator, describes the photon and defines the photon field. Without assuming a Lagrangian or classical law, section 5, *Electromagnetism*, derives the interacting Dirac equation and establishes Maxwell's equations and the Lorentz force law from the minimal interaction in which a Dirac particle emits or absorbs a photon, thereby showing that the physical mass and coupling constant are equal to their bare values in the low energy limit. Section 6, *Finite Quantum Electrodynamics* defines the Dirac propagator and describes the correspondences with causal perturbation theory (Scharf 1989) and with lattice regularization (e.g., Montvay & Münster, 1994). The calculation of Feynman rules is given in appendix H. Section 7, *Conclusion* summarizes the results and discusses implications.

2 Particles

2.1 Dirac Particles

In RQG I, it was shown via Stone's theorem that the probability interpretation requires a first order Schrödinger equation. There is no covariant first order equation for a spin-less particle[1], and, following Dirac (1928), a spin index is added to the ket. When there is no ambiguity spin is suppressed. Explicitly

$$|x\rangle = |x,\alpha\rangle = |x\rangle_\alpha \tag{2.1.1}$$

I will use the normalisation

$$\forall x, y \in \mathscr{D} \quad \langle x,\alpha|y,\beta\rangle = \langle x|y\rangle_{\alpha\beta} = \chi^{-3}\delta_{\boldsymbol{xy}}\delta_{\alpha\beta}\,. \tag{2.1.2}$$

The inner product is

$$\langle g|f\rangle = \chi^3\sum_{\mathrm{D}}\langle g|x\rangle\langle x|f\rangle = \chi^3\sum_{\mathrm{D}} g_\alpha(x)^\dagger f_\alpha(x), \tag{2.1.3}$$

where the summation convention is used for spin indices. As described in RQG I, position functions defined on the discrete spacetime coordinate system S are embedded into smooth wave functions. Wave functions now have a spin index,

$$f_\alpha(x)\big|_{\mathrm{S}} = \langle x|f\rangle_\alpha\,, \tag{2.1.4}$$

The first order equation required by Stone's theorem is the Dirac equation

$$i\partial\cdot\gamma f(x) = mf(x)\,. \tag{2.1.5}$$

Using bold type for 3-vectors, the solution to the Dirac equation is (appendix C)

$$f_\alpha(x) = \left(\tfrac{1}{2\pi}\right)^{3/2}\sum_{r=1}^{2}\int_{\mathrm{M}} d^3\boldsymbol{p}\; F(\boldsymbol{p},r)u_\alpha(\boldsymbol{p},r)\; e^{-ix\cdot p}, \tag{2.1.6}$$

where $F(\boldsymbol{p},r)$ is the momentum space wave function given at $x_0 = 0$ by

$$F(\boldsymbol{p},r) = \langle\boldsymbol{p},r|f\rangle = \left(\tfrac{1}{2\pi}\right)^{3/2}\chi^3\sum_{\mathrm{D}} u(\boldsymbol{p},r)^\dagger f(0,\boldsymbol{x})e^{ix\cdot\boldsymbol{p}}\,, \tag{2.1.7}$$

p satisfies the mass shell condition, and u is a Dirac spinor having the form,

$$u(\boldsymbol{p},r) = \sqrt{\frac{p_0+m}{2p_0}}\begin{bmatrix}\zeta(r)\\ \dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{p_0+m}\zeta(r)\end{bmatrix} \qquad \text{for } r = 1,2, \tag{2.1.8}$$

where $\boldsymbol{\sigma} = (\sigma_1,\sigma_2,\sigma_3)$ are the Pauli spin matrices and ζ is a two-spinor normalised so that

$$\zeta_\alpha(r)^\dagger\zeta_\alpha(s) = \delta_{rs}\,, \tag{2.1.9}$$

where the summation convention is used for repeated spin indices. In this normalisation $u_a(\boldsymbol{p},r)^\dagger u_\alpha(\boldsymbol{p},s) = \delta_{rs}$ (appendix D). It is common to choose a relativistic normalisation by multiplying Dirac spinors by $\sqrt{2p^0}$. Since we ultimately divide by normalisation to calculate probability, this makes no difference to predictions. The normalisation used here is consistent with the idea that probability is observer dependent, and leads to simpler formulae.

2.2 Antiparticles

The treatment of the antiparticle modifies the Stückelberg-Feynman (1941 & 1949 respectively) interpretation by considering the mass shell condition. The Dirac equation is most readily understood as the equation of motion for a particle in its own proper time. If every particle has its own proper time. If there is no other fundamental time, then it is natural to think that one particle's proper time can be reversed compared to another; anti-

1. This applies to fundamental particles but does not preclude scalar composite or scalar ghost particles.

matter is matter whose proper time is inverted compared to surrounding matter. A sign is lost in the mass shell condition, due to the squared terms, but a time-like vector with a negative time-like component provides a natural definition of $m < 0$. So, permissible solutions of the Dirac equation, (2.1.5), have positive energy $E = p^0 > 0$ when m is positive and negative energy when m is negative. Complex conjugation reverses time, and the direction of momentum, while maintaining the probability interpretation and restores positive energy, and we also change the sign of mass, $m \to -m$. Thus the negative energy solution is transformed and satisfies

$$i\partial \cdot \bar{\gamma} f(x) = -mf(x), \tag{2.2.1}$$

where $\bar{\gamma}$ is the complex conjugate, $\bar{\gamma}^j_{\alpha\beta} = \overline{\gamma^j_{\alpha\beta}}$. The solution is the wave function for the antiparticle

$$f(x) = \left(\tfrac{1}{2\pi}\right)^{3/2} \sum_{r=1}^{2} \int_M d^3\boldsymbol{p}\ F(\boldsymbol{p}, r)\bar{v}(\boldsymbol{p}, r)\ e^{-ix\cdot p}, \tag{2.2.2}$$

where $F(\boldsymbol{p},r)$ is the momentum space wave function given by

$$F(\boldsymbol{p}, r) = \left(\tfrac{1}{2\pi}\right)^{3/2} \chi^3 \sum_D \bar{v}(\boldsymbol{p}, r)^\dagger f(0, \boldsymbol{x}) e^{ix\cdot \boldsymbol{p}}\ , \tag{2.2.3}$$

p satisfies the mass shell condition, and $\bar{v}$ is the complex conjugate of the Dirac spinor,

$$v(\boldsymbol{p}, r) = \sqrt{\frac{p_0 + m}{2p_0}} \begin{bmatrix} \dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{p_0 + m}\zeta(r) \\ \zeta(r) \end{bmatrix} \qquad \text{for } r = 1,2. \tag{2.2.4}$$

The spinor has the normalisation $\bar{v}_\alpha(\boldsymbol{p}, r)^\dagger \bar{v}_\alpha(\boldsymbol{p}, s) = \delta_{rs}$. The use of $\bar{v}$ in (2.2.2) and (2.2.3) will be reconciled with the more usual v in the definition of the field operators.

2.3 Conserved Current

Definition: The **Dirac adjoint** of a Dirac spinor, u, is $\widehat{u} = u^\dagger\gamma^0$. .

Since the interpretation is based on probability theory, we need a relativistic statement that probability is conserved. That is, we require a vector current density j^a such that

$$\partial_a j^a = 0 \tag{2.3.1}$$

and the probability density for finding a particle at x is

$$j^0(x) = f(x)^\dagger f(x) \tag{2.3.2}$$

Postulate: Current density is $j^a = \widehat{f}\gamma^a f$.

Current density satisfies,

$$\begin{aligned} \partial_a j^a &= \partial_a (f^\dagger\gamma^0\gamma^a f) = (\partial_a f^\dagger)\gamma^{a\dagger}\gamma^0 f + \widehat{f}\gamma^a\partial_a f \\ &= im\,\widehat{f}f - im\,\widehat{f}f = 0\,. \end{aligned} \tag{2.3.3}$$

as is required of a conserved current.

2.4 Creation Operators

In interactions, particles may be created and destroyed. The creation of a particle in an interaction is described by the action of a creation operator, and destruction is described by an annihilation operator. A change of state of a particle can be described as the annihilation of one state and the creation of another. Thus, a complete description of any process in interaction can be achieved through combinations of creation and annihilation operators. Creation and annihilation operators are linear operators, and incorporate the idea that when a particle is created it is impossible to distinguish it from any existing particle of the same type, so that they automatically (anti)symmetrise states of identical particles. A creation operator is closely associated with the state which it creates, and will be denoted as a ket, with an underline to distinguish it from a state.

Definition: $\forall x \in \mathrm{D}$ the **creation operator** $|\underline{x}\rangle : \mathbb{H} \to \mathbb{H}$ is $\forall y,\ y^i \in \mathrm{N}, i = 1, \ldots, n$

$$|\underline{x}\rangle|\,\rangle = |x\rangle$$

$$|\underline{x}\rangle : |y\rangle \to |x\rangle|y\rangle = |x;y\rangle = \tfrac{1}{\sqrt{2}}[(|x\rangle, |y\rangle) + \kappa(|y\rangle, |x\rangle)]$$

$$|\underline{x}\rangle : |y^1\rangle\ldots|y^n\rangle \to \frac{1}{\sqrt{n+1}}\left(\sum_{i=0}^{n} \kappa^i |y^1\rangle\ldots|y^i\rangle|x\rangle|y^{i+1}\rangle\ldots|y^n\rangle\right) \tag{2.4.1}$$

where $|x\rangle$ appears in the *i*+1th position in the *i*th term of the sum. It is routine to show that $\kappa = \pm 1$ for Bosons and Fermions respectively. More generally, creation operators are defined by linearity

$$\forall\, |f\rangle \in \mathbb{H}^1 \qquad |\underline{f}\rangle = \chi^3 \sum_{x \in \mathrm{D}} \langle x|f\rangle |\underline{x}\rangle \tag{2.4.2}$$

The space of (anti)symmetric states $\mathbb{F} \subset \mathbb{H}$ is generated from $\mathbb{H}^0 = \{|\,\rangle\}$ by creation operators. Physical states are elements of $\mathbb{F}$.

Definition: $\forall |f\rangle \in \mathbb{H}^1$, the **annihilation operator** $\langle \underline{f}| : \mathbb{F} \to \mathbb{F}$ is the Hermitian conjugate of the creation operator $|\underline{f}\rangle : \mathbb{F} \to \mathbb{F}$, $\langle \underline{f}| = |\underline{f}\rangle^{\dagger}$.

3 Interactions

3.1 The Interaction Hamiltonian

Time evolution is modelled by a continuous operator, $U(t) : \mathbb{F} \to \mathbb{F}$, such that $U(t)$ is formally the same for all t_0 (RQG I section 5),

$$U(t) = e^{-iHt} \tag{3.1.1}$$

Postulate: In a time interval, *t*, there either is, or is not, an interaction. By the identification of the operations of vector space with weighted OR between uncertain possibilities, time evolution including the possibility of an interaction is described by a **Hamiltonian**, $H : \mathbb{F} \to \mathbb{F}$, with

$$H = H_0 + H_{\mathrm{int}} \tag{3.1.2}$$

where H_0 is the **free Hamiltonian**, and H_{int} is a Hermitian operator describing an interaction between particles, called the **interaction Hamiltonian**. H_{int} is defined with no component corresponding to the absence of interaction,

$$\forall \boldsymbol{x}^i \in \mathrm{N}, \forall n \in \mathbb{N},\ \langle \boldsymbol{x}^1;\ldots;\boldsymbol{x}^n|H_{\text{int}}|\boldsymbol{x}^1;\ldots;\boldsymbol{x}^n\rangle = 0\ , \tag{3.1.3}$$

In general, H_{int} will be a sum of terms for different types of interaction. Here only one type of interaction will be considered. Thus, the evolution of a state is given by

$$\partial_0|f(t)\rangle = -iH|f(t)\rangle = -i(H_0+H_{\text{int}})|f(t)\rangle \tag{3.1.4}$$

It is convenient to separate interactions from free particle evolution by working in the interaction picture, so that

$$|f_{\mathrm{I}}(t)\rangle = e^{iH_0t}|f(t)\rangle \tag{3.1.5}$$

$$A_{\mathrm{I}}(t) = e^{iH_0t}Ae^{-iH_0t} \tag{3.1.6}$$

$$H_{\mathrm{I}}(t) = e^{iH_0t}H_{\text{int}}e^{-iH_0t} \tag{3.1.7}$$

As is common practice, the suffix I denoting the interaction picture will be dropped when there is no ambiguity. Evolution is given in the interaction picture by

$$U(t) = e^{iH_0t}e^{-iHt} \tag{3.1.8}$$

Differentiating gives (appendix A)

$$\dot{U}(t) = -iH_{\mathrm{I}}U(t) \tag{3.1.9}$$

which has solution ($U(0) = 1$)

$$U(t) = e^{-iH_{\mathrm{I}}t} \tag{3.1.10}$$

3.2 The Hamiltonian Density

We assume that we can define a Hermitian interaction density operator, $I(x)$, having the same effect on a matter anywhere and at any time, as required by the general principle of relativity. By the identification of addition with logical disjunction, H_{I} can be written as a sum.

Postulate: The **Hamiltonian (or interaction) density**, $I(x) : \mathbb{F} \to \mathbb{F}$, is a Hermitian operator such that the interaction Hamiltonian is

$$H_{\mathrm{I}}(x^0) = \chi^3\sum_{\mathscr{D}} I(x)\ . \tag{3.2.1}$$

3.3 The Perturbation Expansion

Without loss of generality, let $t_0 = 0$. Let $t_j = j\chi$ for $j \in \mathbb{Z}$. Then the discrete time interval is $\mathscr{T} = \{t_j | 0 \le j \le n \text{ for some } n \in \mathbb{Z}\}$

$$U(t_{j+1}) = U(\chi)U(t_j) = (1 - iH_{\mathrm{I}}(t_j)\chi)U(t_j) \tag{3.3.1}$$

Iterating

$$U(t_1) = 1 - iH_{\mathrm{I}}(t_0)\chi \tag{3.3.2}$$

$$U(t_2) = (1 - iH_{\mathrm{I}}(t_1)\chi)(1 - iH_{\mathrm{I}}(t_0)\chi) \tag{3.3.3}$$

Expand after n iterations

$$U(t_n) = 1 + (-i\chi)\sum_{i=0}^{n-1} H_{\mathrm{I}}(t_i) + (-i\chi)^2 \sum_{j>i}\sum_{i=0}^{n-1} H_{\mathrm{I}}(t_j)H_{\mathrm{I}}(t_i) + \ldots . \tag{3.3.4}$$

Substituting (3.2.1)

$$U(t_n) = 1 + (-i\chi^4)\sum_{i=0}^{n-1} I(x_i) + (-i\chi^4)^2 \sum_{j>i}\sum_{i=0}^{n-1} I(x_j)I(x_i) + \ldots , \tag{3.3.5}$$

where the sums are now over space as well as time. This can be rewritten

$$U(t) = 1 + \sum_{k=1}^{n} (-i\chi^4)^k \sum_{i_k > i_{k-1}} \ldots \sum_{i_2 > i_1} \sum_{i_1 = 1}^{n} I(x_{i_k})\ldots I(x_{i_2}) I(x_{i_1})$$

. (3.3.6)

It will be observed that in the limit $\chi \to 0$ this reduces to the standard integral form (appendix E), and that improper integrals must be used. In consequence care is needed in the order of taking limits.

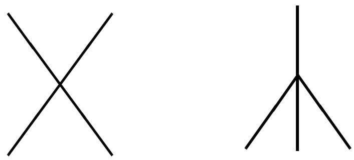

Figure 1: Interaction terms showing a) two creation and two annihilation operators, and b) one creation and three annihilation operators.

3.4 Time Ordered Diagrams

Any operator on Fock space, $\mathbb{F}$, can be written as a sum of products of creation and annihilation operators. The change of state associated with an interaction can be described as the annihilation of one state and the creation of another. Thus, a complete description of any process in interaction can be achieved through combinations of creation and annihilation operators. Expand the interaction density, $I(x)$, as a sum of terms of the form

$$i(x) = |\underline{x}\rangle_1 \ldots |\underline{x}\rangle_m \langle \underline{x}|_{m+1} \ldots \langle \underline{x}|_n \tag{3.4.1}$$

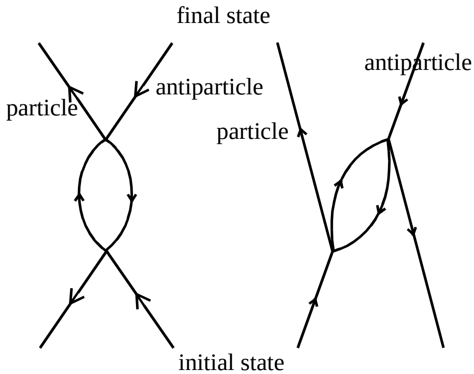

Figure 2: Time-ordered diagrams for two interactions

where $|\underline{x}\rangle_i$ and $\langle \underline{x}|_i$ are creation and annihilation operators for the particles and antiparticles in the interaction. $i(x)$ can be represented diagrammatically as a vertex or node (figure 1). The diagram is time ordered from bottom to top so that the lines above the node correspond to creation operators, and those below the node correspond to annihilation operators

The perturbation expansion for $\langle g(t)|U(t)|f(0)\rangle$ generates a braket between each annihilation operator, $\langle \underline{x}|_i$, and every earlier creation operator, $|\underline{x}\rangle_j$, and every particle in $|f(0)\rangle$, and a braket between every creation operator, $|\underline{x}\rangle_i$, and every particle in the final state, $\langle g|$, All other brakets are zero. These brakets can be represented graphically by connecting corresponding vertices (figure 2). Lines representing particles are shown with arrows from bottom to top, and lines representing antiparticles with arrows from top to bottom. Then the n^{th} term of the perturbation expansion is a sum of terms, each represented as a time-ordered graph containing n vertices.

3.5 The Locality Condition

Definition: Let t_j $\{u_j | u_j = t_k \text{ for some } t_k \in \mathscr{T}\}$ be an unordered set of times in T. Let π be the permutation such that $t\pi_{(1)} > t\pi_{(2)} > \ldots t\pi_{(n)}$. Then the **time-ordered product** is

$$\mathrm{T}\{I(t_1)\ldots I(t_n)\} = I(t_{\pi(1)})\ldots I(t_{\pi(n)}) \tag{3.5.1}$$

Hence, we can write the perturbation expansion (3.3.6)

$$U(t_n) = 1 + \sum_{k=1}^{n} \frac{(-i\chi^4)^k}{k!} \sum_{i_k \neq i_{k-1}, i_k, \ldots, i_1} \cdots \sum_{i_2 \neq i_1} \sum_{i_1 = 1}^{n} \mathrm{T}\{I(x_{i_k})\ldots I(x_{i_2}) I(x_{i_1})\}. \tag{3.5.2}$$

Theorem: (*Locality*) For any x, y, such that $x - y$ is space-like, $[I(y), I(x)] = 0$.

Proof: Under Lorentz transformation, the order of interactions, $I(x_i)$, $I(x_j)$ can be changed in the time-ordered product whenever $x_i - x_j$ is space-like. Under the condition that the initial and final states are stable states of free particles, as in scattering experiments, the calculation of probabilities cannot be affected. Locality follows immediately.

4 Field Theory

4.1 The Dirac Field Operator

The interaction of a particle will be modelled by the annihilation of the old state of the particle and the creation of a new state. Since we can measure the position of an electron, it must be possible to form a projection operator for position at given time,

$$X_{\mathrm{P}}(x) = |x\rangle\langle x| \tag{4.1.1}$$

from the Hamiltonian density and a suitable configuration of matter (more accurately, $X_{\mathrm{P}}(x)$ is summed over a small range of positions depending on the resolution of measurement). The interpretation of antiparticles as negative energy particles going backwards in time means that the annihilation of a negative energy particle appears as the creation of a (positive energy) antiparticle, so that antiparticle annihilation operators appear symmetrically in the interaction operator in a sum with creation operators. This motivates the definition of the Dirac field operator.

Definition: The **Dirac field operator** annihilates an electron or creates an antiparticle,

$$\psi_\alpha(x) = |\overline{x}, \alpha\rangle + \langle \underline{x}, \alpha| \tag{4.1.2}$$

Interaction operators will be products of field operators. There are strong reasons, base on locality, for thinking that interaction operators are products of field operators with the same general form. The Hermitian conjugate of a quantum field operator, has the reverse effect, creating a particle or annihilating an antiparticle. The observable quantity, current density, uses the Dirac adjoint, so we expect the Dirac adjoint operators to appear alongside field operators in the Hamiltonian density.

Definition: The **Dirac adjoint** of the annihilation operator $\langle \underline{x}, \alpha|$ creates a particle,

$$|\underline{\widehat{x}}, \alpha\rangle = \sum_\mu |\underline{x}, \mu\rangle \gamma^0_{\mu\alpha} = \left(\tfrac{1}{2\pi}\right)^{3/2} \sum_r \int_{\mathrm{M}} d^3\boldsymbol{p}\, \widehat{u}_\alpha(\boldsymbol{p}, r) e^{ip\cdot x} |p, r\rangle \tag{4.1.3}$$

The bound on the momentum space integral does not affect covariance, since the operators act on states having wave functions with bounded support in momentum space.

Definition: The **Dirac adjoint** of the antiparticle creation operator $|\overline{x}, \alpha\rangle$ annihilates an antiparticle,

$$\langle\widehat{\overline{x}}, \alpha| = \sum_{\mu}\langle\overline{x}, \mu|\gamma^0_{\mu\alpha} = \left(\tfrac{1}{2\pi}\right)^{3/2}\sum_{r}\int_{M} d^3\boldsymbol{p}\,\widehat{v}_{\alpha}(\boldsymbol{p}, r)e^{ip\cdot x}|p, r\rangle\,. \tag{4.1.4}$$

Definition: The **Dirac adjoint** of the field operator creates a particle or annihilates an antiparticle,

$$\widehat{\psi}_{\alpha}(x) = \psi^{\dagger}_{\mu}(x)\gamma^0_{\mu\alpha} = |\underline{x}, \alpha\rangle + \langle\widehat{\overline{x}}, \alpha|\,. \tag{4.1.5}$$

4.2 Locality of Dirac Field Operators

Since Dirac particles are Fermions we have anticommutation relations for the Dirac field operator.

$$\{\psi_{\alpha}(x), \psi_{\beta}(y)\} = \{\widehat{\psi}_{\alpha}(x), \widehat{\psi}_{\beta}(y)\} = 0\,. \tag{4.2.1}$$

Dirac field operators will appear in pairs in the Hamiltonian density in such a way as to ensure commutation relations required of the locality condition.

Theorem: The equal time anticommutation relations for the Dirac field and Dirac adjoint and obey:

$$\{\psi_{\alpha}(x), \widehat{\psi}_{\beta}(y)\}_{x^0=y^0} = \gamma^0_{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{y})\,. \tag{4.2.2}$$

Theorem: (locality) The anticommutation relation for the Dirac field and the Dirac adjoint is zero outside the light cone.

Proof: appendix F

4.3 The Current Density Observable

For electrons, current is an observable quantity. Since measurement is always the result of interactions between matter, a Hermitian operator, *j*, whose expectation is the classical electrical current must appear in the Hamiltonian density. To ensure that locality is satisfied, current is composed of Dirac field operators.

Postulate: The **current density observable** is $j^a(x) = :\widehat{\psi}(x)\gamma^a\psi(x):$

Then, given the particle state $|f\rangle$ in $\mathbb{H}^1$ with $f(x) = \langle x|f\rangle$,

$$\begin{aligned}\langle j^a(x)\rangle &= \langle f|j^a(x)|f\rangle = \langle f|:(|\underline{\overline{x}}\rangle + \langle\widehat{\overline{x}}|)\gamma^a(|\overline{x}\rangle + \langle\underline{x}|):|f\rangle \\ &= \widehat{f}(x)\gamma^a f(x)\,,\end{aligned} \tag{4.3.1}$$

in agreement with current density for a single particle state (2.3.3). For antiparticle states, spin indices are transposed. Transposition is equivalent to pre- and post-multiplying γ by a matrix, with ones in the trailing diagonal and zeros elsewhere. This has the effect of reversing the order of the spin indices. Thus, a negative energy spin down electron will appear as a positive energy spin up positron.

4.4 Photons

We seek to introduce interactions between particles, such that the interaction operator has an invariant form. Since the current density observable, $j^a(x)$, is a vector, a covariant theory can be found by contracting it with another Hermitian vector operator, $A^a(x)$. The possibilities are severely restricted. The natural and simplest thing to try is to introduce a particle with a spin index which transforms as a vector, and which is its own antiparticle, i.e. its creation and annihilation operators appear in the same field operator. Vector particles may have non-zero mass, but empirical evidence is that this is not so for the photon at the limit of experimental accuracy. Zero mass is assumed.

Definition: The **photon field operator** is $A^a(x) = |\underline{x}, a\rangle + \langle\underline{x}, a|$.

Postulate: The **Hamiltonian density for qed** is $I(x) = ej^a(x)A_a(x)$, where e is an experimentally determined constant, **charge**.

With this Hamiltonian density, photons are always either created or destroyed in interaction. We cannot, therefore, talk of measurements of the position of a photon (a position observable), but only of measurement of the position at which it was annihilated, or the position at which it was created. x is not the position of a photon, but rather the position at which a charged particle would be found to have emitted or absorbed a photon, if a measurement were carried out. This requires that we extend the fundamental rules of quantum theory introduced in RQG I.

RULE VIIIa. $|x, a\rangle = |\underline{x}, a\rangle|\,\rangle$ is the formal conditional clause *"If a measurement found the creation of a photon at x, …"*.

RULE VIIIb. $\langle x, a| = \langle\,|\langle\underline{x}, a|$ is the formal consequent clause *"…, then a measurement would find the annihilation of a photon at x"*.

RULE VIIIc. The **photon position function**, $\langle x, a|f\rangle = \langle\,|A^a(x)|f\rangle$ is the formal statement, *"if $|f\rangle$ were known from previous measurement, then, another measurement would find the annihilation of the photon at x"*.

4.5 Plane Wave Photon States

Since momentum is a conserved quantity (appendix H4), it is possible to talk about the measured momentum of a photon state. A photon created with a given momentum will be annihilated with the same momentum. So, it will be required that plane wave states are an orthogonal basis. First define a basis for spin states.

Definition: For momentum $\boldsymbol{p}$,

$\boldsymbol{w}(\boldsymbol{p}, 3)$ is a longitudinal unit 3-vector, $\boldsymbol{w}(\boldsymbol{p}, 3) = \boldsymbol{p} / |\boldsymbol{p}|$,
$\boldsymbol{w}(\boldsymbol{p}, 1)$ and $\boldsymbol{w}(\boldsymbol{p}, 2)$ are orthogonal transverse unit vectors, so that, for $r = 1, 2, 3$, $\boldsymbol{w}(\boldsymbol{p}, r) \cdot \boldsymbol{w}(\boldsymbol{p}, s) = \delta_{rs}$. The normalised *spin vectors* are $w(\boldsymbol{p}, 0) = (1, \mathbf{0})$ and $w(\boldsymbol{p}, r) = (0, \boldsymbol{w}(\boldsymbol{p}, r))$.

Definition: For momentum $\boldsymbol{p}$, the **photon plane wave state**, $|\boldsymbol{p}, r\rangle$, in $\mathbb{H}^1$ is given by the wave function,

$$\langle x|\boldsymbol{p}, r\rangle = \lambda(|\boldsymbol{p}|, r)w(\boldsymbol{p}, r)e^{-ix\cdot p}, \tag{4.5.1}$$

where $p^2 = 0$ (the mass shell condition) and λ is a scalar, to be determined.

The scalar, λ, is required because the states $|x\rangle = |\underline{x}\rangle|\rangle$ refer to the hypothetical measurement of position of the electron which emits a photon, not the position at which a photon can be measured. Since photons are always created or annihilated in interaction, and cannot be in eigenstates of a position operator, we do not require that the states $|x, a\rangle = |\underline{x}, a\rangle|\rangle$, are orthogonal.

Direction is determined by the distribution of matter, not by fundamental assumption, so λ depends only on the magnitude of $\boldsymbol{p}$. We require that $\langle \boldsymbol{q}, s|\boldsymbol{p}, r\rangle$ is a delta function,

$$\langle \boldsymbol{q}, s|\boldsymbol{p}, r\rangle = \eta(r)\delta_{rs}\delta(\boldsymbol{p}-\boldsymbol{q}), \tag{4.5.2}$$

where $\eta(0) = -1$ and $\eta(r) = 1$ for $r = 1, 2, 3$. The minus sign from $\eta(0)$ does not alter the expansion of the inner product for an orthonormal basis. The braket for the photon is,

$$\langle g|f\rangle = \sum_{r=0}^{3}\int_{\mathbb{M}} d^3\boldsymbol{p}\langle g|\boldsymbol{p}, r\rangle\langle \boldsymbol{p}, r|f\rangle. \tag{4.5.3}$$

The resolution of unity takes the form,

$$1 = \sum_{r=0}^{3}\int_{\mathbb{M}} d^3\boldsymbol{p}|\boldsymbol{p}, r\rangle\langle \boldsymbol{p}, r|. \tag{4.5.4}$$

We do not have $\langle f|f\rangle > 0$; the braket is not positive definite, in conflict with the calculation of probabilities. In practice, we only need to generate probabilities for observations. Since probabilities must be positive, we impose the condition that, in observations on the photon, there is no polarisation between time-like and longitudinal states,

$$\langle \boldsymbol{p}, 0|f\rangle = \langle \boldsymbol{p}, 3|f\rangle. \tag{4.5.5}$$

Physically, any polarisation breaking (4.5.5) would need to be caused by an interaction creating that polarisation. Clearly, no such interaction is known or possible. If one were to start with Coulomb gauge (as used in many standard approaches), then (4.5.5) is automatically satisfied after Lorentz transform. With this restriction, probabilities for the observation time-like and longitudinal states are zero, and the braket reduces to

$$\langle g|f\rangle = \sum_{r=1}^{2}\int_{\mathbb{M}} d^3\boldsymbol{p}\langle g|\boldsymbol{p}, r\rangle\langle \boldsymbol{p}, r|f\rangle. \tag{4.5.6}$$

which is positive semidefinite. It will be seen that all four polarisation states are required for the derivation of Maxwell's equations. We can conclude that the unobservable states have a real effect, and represent real particles, but the probability interpretation allows only the observation of a subspace containing the two transverse polarisation states, on which the inner product is positive definite. The braket is invariant under the addition of a light-like polarisation state, from which it follows that light-like polarisation cannot be determined from experimental results.

We require that the probability for the creation of a photon at x and its annihilation at y is invariant. Observe that

$$\sum_{r=0}^{3}\eta(r)w^a(\boldsymbol{p}, r)w^b(\boldsymbol{p}, r) = -g^{ab}. \tag{4.5.7}$$

Then, setting

$$\lambda(|\boldsymbol{p}|, r) = \left(\tfrac{1}{2\pi}\right)^{3/2} \frac{1}{\sqrt{2p^0}} \tag{4.5.8}$$

gives

$$\begin{aligned}\langle \underline{x}, a | \underline{y}, b\rangle &= \sum_{r=0}^{3} \int_{M} d^3\boldsymbol{p} \langle \underline{x}, a | \boldsymbol{p}, r\rangle\langle \boldsymbol{p}, r | \underline{y}, b\rangle \\ &= \sum_{r=0}^{3} \int_{M} d^3\boldsymbol{p}\, \eta(r)\lambda(|\boldsymbol{p}|, r)\lambda(|\boldsymbol{p}|, r) w^a(\boldsymbol{p}, r) w^b(\boldsymbol{p}, r) e^{-ip\cdot(x-y)} \\ &= -\frac{g^{ab}}{8\pi^3}\int_{M} \frac{d^3\boldsymbol{p}}{2p^0} e^{-ip\cdot(x-y)} \\ &= -\frac{g^{ab}}{8\pi^3}\int_{M} d^4p\, e^{-ip\cdot(x-y)}\delta(p^2) \ ,\end{aligned} \tag{4.5.9}$$

which is invariant because the bound on the momentum space integral is invariant in Lorentz transformation.

4.6 Evolution of Photon States

We may expand $|x, a\rangle$ in terms of wave states,

$$\begin{aligned}|x, a\rangle &= \sum_{r=0}^{3} \int_{M} d^3\boldsymbol{p} |\boldsymbol{p}, r\rangle\langle \boldsymbol{p}, r | x, a\rangle \\ &= \left(\tfrac{1}{2\pi}\right)^{3/2} \sum_{r=0}^{3} \int_{M} \frac{d^3\boldsymbol{p}}{\sqrt{2p^0}} w^a(\boldsymbol{p}, r) e^{ix\cdot p} |\boldsymbol{p}, r\rangle\end{aligned} \tag{4.6.1}$$

Then the wave function for the state $|f\rangle$ is

$$f^a(x) = \left(\tfrac{1}{2\pi}\right)^{3/2} \sum_{r=0}^{3} \int_{M} \frac{d^3\boldsymbol{p}}{\sqrt{2p^0}} w^a(\boldsymbol{p}, r) e^{-ix\cdot p} \langle \boldsymbol{p}, r | f\rangle \tag{4.6.2}$$

Since p is the momentum vector for a zero mass particle, the wave function satisfies a Klein-Gordon Equation,

$$\partial^2 f^a = 0 . \tag{4.6.3}$$

Conservation of probability applies to the creation and annihilation of particles. Differentiating the wave function gives a first order equation as required by Stone's theorem,

$$\partial_a f^a = 0 . \tag{4.6.4}$$

4.7 The Photon Field Operator

The creation operator for a plane wave state is given by $|\underline{\boldsymbol{p}}, r\rangle|\,\rangle = |\boldsymbol{p}, r\rangle$. Substituting gives the **photon field operator**,

$$\begin{aligned}A^a(x) &= |\underline{x}, a\rangle + \langle \underline{x}, a| \\ &= \sum_{r=0}^{3} \eta(r) \int_{M} \frac{d^3\boldsymbol{p}}{\sqrt{2p^0}} (e^{ip\cdot x} |\underline{\boldsymbol{p}}, r\rangle + e^{-ip\cdot x}\langle \underline{\boldsymbol{p}}, r|) w^a(\boldsymbol{p}, r)\end{aligned} \tag{4.7.1}$$

Theorem: The photon field satisfies $\partial^2 A^a = 0$.

Proof: Differentiate A^a twice.

Theorem: For physical states, the photon field satisfies the Gupta-Bleuler gauge condition, $\partial_a A^a | f\rangle = 0$.

Proof: Differentiate A^a and use absence of polarisation between light-like and longitudinal states.

4.8 The Locality Condition for Photons

Photons are Bosons, obeying commutation relations,

$$\begin{aligned}[A^a(x), A^a(y)] &= [|\underline{x}, a\rangle + \langle \underline{x}, a|, |\underline{y}, b\rangle + \langle \underline{y}, b|] \\ &= \langle \underline{x}, a | \underline{y}, b\rangle - \langle \underline{y}, b | \underline{x}, a\rangle \\ &= -\frac{g^{ab}}{8\pi^3}\int_M \frac{d^3\boldsymbol{p}}{2p^0}(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}).\end{aligned} \tag{4.8.1}$$

Substituting $p \to -p$ in the second term gives the equal time commutator,

$$[A(x), A(y)]_{x^0 = y^0} = 0\,. \tag{4.8.2}$$

Because the photon commutator vanishes, the time evolution of the expectation of the photon field is trivial. Physical laws depend on derivatives of the photon field.

Theorem: The equal time commutation relations for the photon field and its derivative obey:

$$[\partial_i A^a(x), A^b(y)]_{x^0 = y^0} = -i\delta_i^0 g^{ab}\delta(\boldsymbol{x} - \boldsymbol{y})\,. \tag{4.8.3}$$

Theorem: (locality) The commutator for the photon field and its derivative is zero outside the light cone.

Proof: Differentiating,

$$\frac{\partial}{\partial x}\langle \underline{x}, a| \underline{y}, b\rangle = -\left(\tfrac{1}{2\pi}\right)^3 g^{ab}\int_M \frac{d^3\boldsymbol{p}}{2p^0} ip e^{-ip\cdot(x-y)}\,, \tag{4.8.4}$$

and

$$\frac{\partial}{\partial x}\langle \underline{y}, b|\underline{x}, a\rangle = \left(\tfrac{1}{2\pi}\right)^3 g^{ab}\int_M \frac{d^3\boldsymbol{p}}{2p^0} ip e^{ip\cdot(x-y)}\,. \tag{4.8.5}$$

Substitute $\boldsymbol{p} \to -\boldsymbol{p}$ at $x_0 = y_0$. Then, for $i = 1, 2, 3$,

$$[\partial_i A^a(x), A^b(y)]_{x^0 = y^0} = \frac{\partial}{\partial x^i}\langle \underline{x}, a| \underline{y}, b\rangle - \frac{\partial}{\partial x^i}\langle \underline{y}, b|\underline{x}, a\rangle = 0\,, \tag{4.8.6}$$

and, for the time-like component,

$$[\partial_0 A^a(x), A^b(y)]_{x^0 = y^0} = -i\left(\tfrac{1}{2\pi}\right)^3 g^{ab}\int_M d^3\boldsymbol{p}\, e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} = -ig^{ab}\delta(x - y)\,. \tag{4.8.7}$$

The integrals are invariant (see RQG I section 6, *Quantum Covariance*), so they are zero outside the light cone.

5 Electromagnetism

5.1 The interaction density for QED

Observable quantities are determined through measurement, i.e. through interaction with other matter. The photon field operator, $A(x)$, is Hermitian. If $\langle A(x)\rangle$ is a classical quantity, $A(x)$ must appear in the Hamiltonian density. QED uses the intuitively appealing minimal interaction, in which single photons are emitted and absorbed by electrons.

Postulate: Interactions are described by the Hamiltonian density,

$$I(x) = ej^a(x)A_a(x)\,, \tag{5.1.1}$$

where j is the current density observable, $j^a(x) = :\overparen{\psi}(x)\gamma^a\psi(x):$ (section 4.3).

To establish that $\langle A(x)\rangle$ is the classical electromagnetic field, it is necessary to establish the Lorentz force law and Maxwell's equations.

Theorem: $\langle A(x)\rangle$ satisfies the Lorenz gauge condition, $\partial_a\langle A^a(x)\rangle = 0$.

Proof: Apply Ehrenfest's theorem (appendix B). By locality the equal time commutator is zero. Using the Gupta-Bleuler gauge condition,

$$\partial_a\langle A^a(x)\rangle = \langle\partial_a A^a(x)\rangle = 0 \tag{5.1.2}$$

The Lorenz gauge condition fixes gauge up to the unobservable light-like polarisation. In classical electrodynamics one may choose a different gauge without affecting predictions, but here Lorenz gauge is fixed by the Gupta-Bleuler gauge condition.

5.2 Momentum in the Interacting Theory

In the absence of interactions, there is no issue with local gauge freedom. The phase of an electron wave function is fixed at the point of creation and becomes simply the global symmetry of the one particle theory, in which kets can be multiplied by constant phase without altering their meaning in formal language. When interactions are introduced the result is that the evolution of the wave function does not match the evolution of the field operator which created it, and which is defined on the non-interacting space. A difficulty arises because the momentum observable in the non-interacting theory,

$$P^a = i\sum_{\mathrm{D}}|x\rangle\partial^a\langle x| = i\partial^a \tag{5.2.1}$$

extracts the frequency and wavelength of the wave function. We would like to use Ehrenfest's theorem (appendix B) to calculate the classical force due to the interaction, by differentiating the expectation of momentum,

$$\frac{d}{dt}\langle P^a\rangle = \langle\frac{d}{dt}P^a\rangle + i\langle[H, P^a]\rangle \tag{5.2.2}$$

but states evolve according to the full Hamiltonian, whereas the creation operators are defined on the Fock space of non-interacting particles, and create states obeying the Dirac equation. There is a real phase shift corresponding to change in momentum, which must be distinguished from the arbitrary phase in the definition of field operators.

To ensure that creation operators and states evolve identically, we define the **field picture**, using a simplified form of the Foldy-Wouthuysen transformation (Foldy & Wouthuysen 1950) which ignores spin,

$$| f_{\mathrm{F}}(t)\rangle = e^{-iH_{\mathrm{I}}t}| f(t)\rangle = e^{iH_0 t}| f(0)\rangle . \tag{5.2.3}$$

In the field picture states evolve as in the Schrödinger picture for non-interacting particles. The momentum operator in the field picture is

$$P^a_{\mathrm{F}} = e^{-iH_{\mathrm{I}}t} i\partial^a e^{iH_{\mathrm{I}}t} . \tag{5.2.4}$$

In the semi-classical correspondence, for small t, this may be treated as a perturbation to the evolution of a non-interacting particle, in which the interaction is replaced with an expectation. For a classical particle with position x and velocity $v = \dot{x}$, the classical current is $J = -e\dot{z}$. The expectation of the interaction Hamiltonian is

$$\langle H_{\mathrm{I}}\rangle = J \cdot \langle A(x)\rangle = q\dot{x} \cdot \langle A(x)\rangle = -e\dot{x} \cdot \langle A(x)\rangle . \tag{5.2.5}$$

Replacing the interaction Hamiltonian with its expectation, the momentum operator in the field picture is

$$P^a_{\mathrm{F}} = e^{ie\dot{x} \cdot \langle A(x)\rangle t} i\partial^a e^{-ie\dot{x} \cdot \langle A(x)\rangle t} = i\partial^a - e\langle A^a(x)\rangle . \tag{5.2.6}$$

Thus the expectation, $\langle A^a(x)\rangle$, of the operator which creates and annihilates photons, acts in the manner of a classical vector field, modifying energy and momentum. This is the standard formula for generalised momentum in the presence of a field, but normally it is assumed on phenomenological grounds, whereas here it is been found theoretically. With the replacement of the momentum operator for non-interacting particles with the corresponding operator taking interactions into account, $i\partial^a \to P^a_F = i\partial^a - e\langle A^a(x)\rangle$, the Dirac equation, $(i\gamma^a\partial_a - m) f(x) = 0$ becomes the interacting Dirac equation.

$$(\gamma^a(i\partial_a - e\langle A_a(x)\rangle) - m) f(x) = 0 . \tag{5.2.7}$$

5.3 The Lorentz Force Law

Working in the field picture, we have, from Ehrenfest's theorem (appendix B),

$$\frac{d}{dt}\langle P^a_{\mathrm{F}}\rangle = \langle \frac{d}{dt}P^a_{\mathrm{F}}\rangle + i\langle [H, P^a_{\mathrm{F}}]\rangle . \tag{5.3.1}$$

Using expectations for the interaction, as above, we have

$$H = H_0 + H_I \approx H_0 + \langle H_I\rangle = H_0 - e\dot{x} \cdot \langle A(x)\rangle . \tag{5.3.2}$$

Substituting for H, using (5.2.6), and dropping the suffix F (since expectations are the same in any picture),

$$\begin{aligned}\frac{d}{dt}\langle P^a\rangle &= e\frac{d}{dt}\langle A^a(x)\rangle + i\langle [H_0 - e\dot{x} \cdot \langle A(x)\rangle, i\partial^a - e \cdot \langle A^a(x)\rangle]\rangle \\ &= e\frac{d}{dt}\langle A^a(x)\rangle - e\partial^a(\dot{x} \cdot \langle A(x)\rangle) ,\end{aligned} \tag{5.3.3}$$

where the product rule of differentiation has been used to find the second term. Classical force is defined as the rate of change of momentum, according to Newton's second law;

$$(\mathrm{Force})^a = \frac{d}{d\tau}(\mathrm{momentum})^a . \tag{5.3.4}$$

By general covariance, τ is proper time of the matter on which the force acts, not coordinate time defined by a particular observer. The electromagnetic force on a charged particle will be evaluated in the rest frame of the particle, in which $\dot{x} = (1, 0, 0, 0)$ and the current is $J = -e(1, 0, 0, 0)$. In the rest frame,

$$\partial^0 \langle P^a \rangle = e\partial^0 \langle A^a(x) \rangle - e\partial^a \langle A^0(x) \rangle . \tag{5.3.5}$$

So,

$$(\text{Force})^a = \frac{d}{d\tau} \langle P^a \rangle = J_0(\partial^a \langle A^0(x) \rangle - \partial^0 \langle A^a(x) \rangle) . \tag{5.3.6}$$

After Lorentz transformation, this is

$$(\text{Force})^a = \frac{d}{d\tau} \langle P^a \rangle = J_b(\partial^a \langle A^b(x) \rangle - \partial^b \langle A^a(x) \rangle) = J_b F^{ab} \tag{5.3.7}$$

where F^{ab} is the Faraday tensor. This establishes the Lorentz force law.

5.4 Maxwell's Equations

Theorem: $\langle A(x) \rangle$ satisfies Maxwell's equations in Lorenz gauge:

$$\partial^2 \langle A(x) \rangle = -e \langle j(x) \rangle . \tag{5.4.1}$$

Proof: Differentiating the expectation of the photon field twice, using Ehrenfest's theorem (appendix B),

$$\partial^2 \langle A(x) \rangle = \partial_a \langle \partial^a A(x) \rangle = i \langle [H(x), \partial_0 A(x)] \rangle + \langle \partial^2 A(x) \rangle = i \langle [H(x), \partial_0 A(x)] \rangle .$$

Using the Hamiltonian density, $I(x) = ej(x) \cdot A(x)$ (5.1.1),

$$\partial^2 \langle A(x) \rangle = ie\chi^3 \sum_{y \in \mathrm{D}} \langle [j(y) \cdot A(y), \partial_0 A(x)] \rangle \tag{5.4.2}$$

Maxwell's equations in Lorenz gauge (5.4.1) follow immediately by applying the equal time commutator for photons (4.8.3).

6 Finite Quantum Electrodynamics

6.1 The Feynman Propagator

Definition: The **Feynman propagator**, or **contraction** of $\phi^\dagger(y)$ and $\phi(x)$ is

$$D(x-y) = \begin{cases} 0 & \text{if } x^0 = y^0 \\ \langle \underline{x} | \underline{y} \rangle \Theta(x^0 - y^0) \pm \langle \overline{x} | \overline{y} \rangle \Theta(y^0 - x^0) & \text{otherwise} \end{cases} \tag{6.1.1}$$

where + is used for Bosons and – for Fermions, and Θ is the step function, $\Theta(x) = 0$ if $x \leq 0$, $\Theta(x) = 1$ if $x > 0$.

This form of the propagator may be compared with causal perturbation theory (Scharf 1989), using the method of Epstein and Glaser (1973), in which the step functions are replaced with a C^∞ function switching function. The difference is that here we use a discrete sum whereas causal perturbation theory use a continuous switching function, and while Scharf says (p163) “the switching on and off the interaction is unphysical”, here the equal time propagator is specifically excluded from the perturbation expansion (3.5.2), and can be regarded as a physical constraint meaning that only one interaction takes place for each particle in any instant. The analysis of the origin of ultraviolet divergences is effectively the same as that given in causal perturbation theory and lattice regularization in that the limit is taken after removing the equal point multiplication. A further discussion of the origin of the ultraviolet divergence is given in appendix J.

The photon propagator can be evaluated as shown in appendix I2; for $x^0 \neq y^0$

$$D_F(x_i - x_j) = \Theta(x_i^0 - x_j^0)\langle \underline{x}_i | \underline{x}_j \rangle + \Theta(x_j^0 - x_i^0)\langle \underline{x}_j | \underline{x}_i \rangle^{\mathrm{T}}$$

$$= \frac{-ig}{16\pi^4} \operatorname*{Lim}_{\varepsilon \to 0^+} \int d^4\tilde{p} \frac{e^{-i\tilde{p}\cdot(x_i - x_j)}}{\tilde{p}^2 + 2ip^0\varepsilon + \varepsilon^2}, \tag{6.1.2}$$

where $\tilde{p}^0$ is a dummy variable, $\tilde{p} = (\tilde{p}^0, \tilde{\boldsymbol{p}})$ is a non-vector and $\tilde{\boldsymbol{p}}$ is 3-momentum from j to i; $\tilde{\boldsymbol{p}} = \boldsymbol{p}$ if $x_i^0 > x_j^0$ and $\tilde{\boldsymbol{p}} = -\boldsymbol{p}$ if $x_j^0 > x_i^0$. g is the metric tensor.

Similarly the propagator for a Dirac particle can be evaluated as shown in appendix I3; for $x^0 \neq y^0$

$$S_F(x_i - x_j) = \Theta(x_i^0 - x_j^0)\langle \underline{x}_i | \widehat{\underline{x}_j} \rangle - \Theta(x_j^0 - x_i^0)\langle \widehat{\bar{x}_j} | \bar{x}_i \rangle^{\mathrm{T}}$$

$$= \frac{1}{16\pi^4} \operatorname*{Lim}_{\varepsilon \to 0^+} \int d^4\tilde{p} \frac{(i\tilde{p}\cdot\gamma + m)e^{-i\tilde{p}\cdot(x_i - x_j)}}{\tilde{p}^2 - m^2 + i\varepsilon}. \tag{6.1.3}$$

We may now derive Feynman rules following Dyson's calculation (appendix H), but we observe that the integral form of the perturbation expansion (E1.1) contains improper integrals, and that the limit should not be taken until after calculation of each diagram. An energy cut-off is automatically introduced by a finite lattice, and propagators are modified by setting $D(x - y) = 0$ at $x^0 = y^0$. The most straightforward way to determine the effect of the modification is to consider the non-perturbative solution. This allows us to impose regularisation conditions on the propagator at low energies, that it is independent of lattice spacing χ to first order, and that the renormalised mass and charge adopt their bare values, since the derivations of the Lorentz force law (section 5.3) and Maxwell's equations (section 5.4) show that the bare values are physical values.

6.2 The Landau Pole

The Callan-Symanzik equation is found by taking the infinite sum of an asymptotic series (e.g. Peskin & Schroeder, 1995, eq. 10.27). This can be a good approximation at the energies available in measurement, while breaking down at energies approaching the Landau pole. Neither causal perturbation theory nor lattice regularization remove the Landau pole, but if the minimum discrete unit of time, χ, is a fundamental property of nature, the perturbation expansion (3.5.2) terminates after a finite number, n, terms, and the Landau pole does not appear.

6.3 Interpretation of Feynman diagrams

In standard treatments of qed, Feynman diagrams are regarded merely as aids to calculation, not descriptions of underlying structure. In relational quantum gravity, the perturbation expansion is interpreted directly as a quantum-logical statement, meaning that any number of interactions might be found taking place at any time and any position if we were to do a measurement. The sums in the expansion simply represent OR between possibilities. $H_{\mathrm{I}}(x)$ describes the possibility that an interaction might be anywhere, not a quantized "matter field" which is, in some sense, everywhere. Similarly, Feynman's path integral, or "sum over all paths" has as natural interpretation as a logical OR between the possible paths that might be detected if an experiment could be done to trace the path (*not* that a particle passes through all paths in spacetime; e.g. Feynman 1985).

The perturbation expansion (3.5.2) is a sum of terms representing n interactions. Sum stands for disjunction. So, the meaning of the perturbation expansion is that we cannot say how many interactions take place in any given physical process. Feynman diagrams give a pictorial representation of the same statement; in a particle interpretation, Feynman diagrams also give a pictorial representation of the fundamental structure of matter. We cannot say what the precise configuration of particle interactions in any given instance, but we represent each possible configuration as a graph and sum over the possibilities, using the interpretation of a sum as logical disjunction. Only the topology of lines and vertices is relevant. The paper on which the diagram is drawn has no meaning. Spacetime structure does not appear in Feynman diagrams, except in so far as energy-momentum is four dimensional. Vectors are strictly defined in tangent space, from measurement at the origin. Thus Feynman diagrams describe the fundamental structure of a particulate relational model in which only particles exist and in which other properties, including spacetime geometry, emerge from interactions between particles. Francis (2009b; RQG III) will introduce the physical metric, and show that curvature is directly related to the minimum discrete unit of time, χ, between interactions.

7 Conclusion

Classical electromagnetism and quantum electrodynamics have been shown in a particulate model based on the quantum covariant formulation of relativistic quantum mechanics given in RQG I. In a change of coordinate system momentum is a vector, but the inner product is defined on an invariant lattice. The inner product is invariant because it is determined from the measurement apparatus used to define the reference frame. Quantum covariance does not require the limit of small lattice spacing or large lattice size. It is required that terms dependent on lattice spacing are negligible in the predictions of the theory. The Landau pole appears in the limit as the discrete time interval tends to zero, but not if there exists a fundamental interval of proper time between the interactions of a particle.

Using finite dimensional Hilbert space, fields are operators, not distributions, and there is no problem of principle in taking products of fields. However, in a discrete model the equal time product $\phi^\dagger(x)\phi(x)$ does not appear in the perturbation expansion (3.5.2). The origin of the ultraviolet divergence in the integral form of the perturbation expansion (E1.1) is the incorrect order of taking limits for diagrams containing improper integrals; a cut-off must be used and the limit must be taken *after* calculation of each diagram. Provided that limits are not taken prematurely, terms containing the equal point multiplication do not appear in the perturbation expansion. The exclusion of these terms removes cut-off dependencies to first order and regularizes the perturbation expansion.

The regularization condition is that charge and mass adopt their physical values, as in standard treatments. However it is seen from the derivations of the Lorentz force law (section 5.3) and Maxwell's equations (section 5.4) that bare mass and charge are the physical values.

In RQG I quantum theory was not found using the quantization of classical quantities or the second quantization of classical fields, but by making formal statements about hypothetical measurement results. In this interpretation, QED is fundamentally a theory of particles, not a theory of fields. Classical quantities are understood as expectations, describing the large scale behaviour of systems of many particles. Wave functions are statements in the subjunctive move describing the possible positions where a particle might be found if an experiment were done. The photon is also a particle, but since a photon cannot be detected without being annihilated, there is no position operator for a photon. and the photon wave function describes possibilities for where a photon may be found to have been annihilated, not where it is. Likewise, field operators describe possibilities rather than actualities and are the mathematical building blocks for the description of interactions between fundamental particles.

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Appendices

Appendix A The Derivative of *U*

Differentiate directly,

$$\begin{aligned}\dot{U}(t) &= ie^{iH_0t}H_0e^{-iHt} - ie^{iH_0t}He^{-iHt}\\ &= -ie^{iH_0t}H_{\mathrm{int}}e^{-iHt}\\ &= -ie^{iH_0t}H_{\mathrm{int}}e^{-iH_0t}e^{iH_0t}e^{-iHt}\\ &= -iH_{\mathrm{I}}(t)U(t)\,. \end{aligned}\tag{A1.1}$$

Appendix B Ehrenfest's theorem

Theorem: Ehrenfest's Theorem states that

$$\partial_0\langle A\rangle = \langle\partial_0 A\rangle - i\langle[A, H]\rangle\,. \tag{B1.1}$$

Proof: We have (from Stone's theorem), for any $|\,f\,\rangle$,

$$\partial_0|\,f\,\rangle = -iH|\,f\,\rangle$$

Since H is Hermitian,

$$\partial_0^{\dagger}|\,f\,\rangle = iH|\,f\,\rangle$$

Hence,

$$\langle\,f\,|\partial_0 = \langle\,f\,|(iH)$$

where the operators act to the left. Differentiate $\langle A\rangle$ using the product rule,

$$\begin{aligned}\partial_0\langle\,f\,|A|\,f\,\rangle &= (\langle\,f\,|\partial_0)A|\,f\,\rangle + \langle\,f\,|(\partial_0 A)|\,f\,\rangle + \langle\,f\,|A(\partial_0|\,f\,\rangle)\\ &= i\langle\,f\,|HA|\,f\,\rangle + \langle\,f\,|(\partial_0 A)|\,f\,\rangle - i\langle\,f\,|AH|\,f\,\rangle\end{aligned}$$

which establishes Ehrenfest's theorem and governs the classical behaviour of matter.

Corollary: For an observable quantity with no explicit time dependence,

$$\partial_0\langle A\rangle = -i\langle[A, H]\rangle \tag{B1.2}$$

Theorem: For the space indices, $a = 1, 2, 3$, $\partial_a\langle A\rangle = \langle\partial_a A\rangle$.

Proof: Space translation is the same for an observable operator, $A(x)$, and the corresponding classical observable, $A_{\mathrm{c}}(x) = \langle A(x)\rangle$, we have

$$A_{\mathrm{c}}(x - a) = \langle A(x - a)\rangle\,,$$

and hence, differentiating from first principles,

$$\begin{aligned}\partial_a\langle A(x)\rangle &= \lim_{dx^a\to 0}\frac{\langle A(x+dx)\rangle - \langle A(x)\rangle}{dx^a}\\ &= \lim_{dx^a\to 0}\frac{\langle A(x+dx) - A(x)\rangle}{dx^a}\\ &= \langle\partial_a A(x)\rangle\,.\end{aligned}$$

Appendix C Solution of the Dirac Equation

The positive energy solutions to the Dirac equation are

$$\langle x, \mu | f \rangle = f_\mu(x) = \left(\tfrac{1}{2\pi}\right)^{3/2} \sum_{r=1}^{2} \int d^3\boldsymbol{p}\ F(\boldsymbol{p}, r) u_\mu(\boldsymbol{p}, r) e^{-ix \cdot p} \qquad \text{(C1.1)}$$

where p satisfies the mass shell condition and u is a Dirac spinor having the form, for $r = 1,2$, and for a two-spinor ζ normalised so that $\zeta(r)^\dagger \zeta(s) = \delta_{rs}$,

$$u(p, r) = \sqrt{\frac{p^0 + m}{2p^0}} \begin{bmatrix} \zeta(r) \\ \dfrac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{p^0 + m} \zeta(r) \end{bmatrix}.$$

Proof: Observe that

$$\boldsymbol{\sigma} \cdot \boldsymbol{p} = \begin{bmatrix} p^3 & p^1 - ip^2 \\ p^1 + ip^2 & -p^3 \end{bmatrix}$$

So,

$$\begin{aligned} (\boldsymbol{\sigma} \cdot \boldsymbol{p})^2 &= \begin{bmatrix} p^3 & p^1 - ip^2 \\ p^1 + ip^2 & -p^3 \end{bmatrix} \begin{bmatrix} p^3 & p^1 - ip^2 \\ p^1 + ip^2 & -p^3 \end{bmatrix} \\ &= \begin{bmatrix} (p^3)^2 + (p^1)^2 + (p^2)^2 & 0 \\ 0 & (p^3)^2 + (p^1)^2 + (p^2)^2 \end{bmatrix} \\ &= ((p^0)^2 - m^2) 1_2 \end{aligned} \qquad \text{(C1.2)}$$

Hence,

$$\begin{aligned} p_a \gamma^a_{\mu\nu} u_\nu(\boldsymbol{p}, r) &= \sqrt{\frac{p^0 + m}{2p^0}} \begin{bmatrix} p^0 1_2 & -\boldsymbol{\sigma} \cdot \boldsymbol{p} \\ \boldsymbol{\sigma} \cdot \boldsymbol{p} & -p^0 1_2 \end{bmatrix} \begin{bmatrix} \zeta(r) \\ \dfrac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{p_0 + m} \zeta(r) \end{bmatrix} \\ &= \sqrt{\frac{p_0 + m}{2p^0}} \begin{bmatrix} \left(p^0 1_2 - \dfrac{(\boldsymbol{\sigma} \cdot \boldsymbol{p})^2}{p_0 + m}\right) \zeta(r) \\ \boldsymbol{\sigma} \cdot \boldsymbol{p} \left(\dfrac{p^0 + m - p^0}{p^0 + m}\right) \zeta(r) \end{bmatrix} \\ &= \sqrt{\frac{p_0 + m}{2p^0}} \begin{bmatrix} (p^0 1_2 - (p^0 - m) 1_2) \zeta(r) \\ \boldsymbol{\sigma} \cdot \boldsymbol{p} \left(\dfrac{m}{p^0 + m}\right) \zeta(r) \end{bmatrix} \\ &= m \sqrt{\frac{p_0 + m}{2p^0}} \begin{bmatrix} \zeta(r) \\ \left(\dfrac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{p_0 + m}\right) \zeta(r) \end{bmatrix} \\ &= m u_\nu(\boldsymbol{p}, r) \end{aligned} \qquad \text{(C1.3)}$$

Then, differentiation of the solution of the Dirac equation gives,

$$\begin{aligned} i\partial_a\gamma^a_{\mu\nu} f_\nu(x) &= \left(\tfrac{1}{2\pi}\right)^{3/2}\sum_{r=1}^{2}\int_{\mathbb{R}^3} d^3\boldsymbol{p}\ p_a\gamma^a_{\mu\nu}F(\boldsymbol{p},r)u_\nu(\boldsymbol{p},r)\ e^{-ix\cdot p} \\ &= \left(\tfrac{1}{2\pi}\right)^{3/2}\sum_{r=1}^{2}\int_{\mathbb{R}^3} d^3\boldsymbol{p}\ mF(\boldsymbol{p},r)u_\nu(\boldsymbol{p},r)\ e^{-ix\cdot p} \\ &= mf_\nu(x) \end{aligned}$$

as required. The analysis is similar for antiparticles.

Appendix D Normalisation of Dirac Spinors

The Pauli spin matrices are Hermitian. So, using (C1.2),

$$\boldsymbol{\sigma}\cdot\boldsymbol{p}^\dagger\boldsymbol{\sigma}\cdot\boldsymbol{p} = (\boldsymbol{\sigma}\cdot\boldsymbol{p})^2 = ((p^0)^2 - m^2)1_2$$

Then

$$\begin{aligned} u(\boldsymbol{p},r)^\dagger u(\boldsymbol{p},s) &= \frac{p_0+m}{2p^0}\begin{bmatrix}\zeta(r)^\dagger & \zeta(r)^\dagger\dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{p_0+m}^\dagger\end{bmatrix}\begin{bmatrix}\zeta(s) \\ \dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{p_0+m}\zeta(s)\end{bmatrix} \\ &= \frac{p_0+m}{2p^0}\left(1+\frac{p^0-m}{p^0+m}\right)\delta_{rs} \\ &= \delta_{rs}\,. \end{aligned}$$

Appendix E Integral Form of the Perturbation Expansion

If the minimal proper time χ between interactions is small then (3.1.9) will be a good approximation. Integrate directly,

$$U(t) \approx 1 - i\int_0^t dt_1 H_{\mathrm{I}}(t_1)U(t_1)$$

Substituting U iteratively back into the integral gives the Dyson expansion,

$$U(t) \approx 1 + (-i)\int_0^t dt_1 H_{\mathrm{I}}(t_1) + (-i)^2\int_0^t dt_1\int_0^{t_1} dt_2 H_{\mathrm{I}}(t_1)H_{\mathrm{I}}(t_2) + \ldots$$

This can also be verified by differentiating. Each term is the derivative of the next multiplied by -$iH(t)$. Substituting

$$H_{\mathrm{I}}(x_i^0) = \chi^3\sum_{\mathscr{D}} I(x) \approx \int d^3x_i I(x_i)$$

gives

$$U(t) \approx 1 + \sum_{k\geq 1}(-i)^k \int_{x_1^0<t} d^4x_1 \int_{x_2^0<x_1^0} d^4x_2 \ldots \int_{x_k^0<x_{k-1}^0} d^4x_k I(x_1)I(x_2)\ldots I(x_k) \qquad \text{(E1.1)}$$

It can be seen that, provided that the integrals are defined,

$$\int d^4x_1 \int_{x_2^0 < x_1^0} d^4x_2 \ldots \int_{x_n^0 < x_{n-1}^0} d^4x_n I(x_1)\ldots I(x_n) = \frac{1}{n!}\int d^4x_1 \int d^4x_2 \ldots \int d^4x_n T\{I(x_1)\ldots I(x_n)\}$$

Hence, we can write the perturbation expansion

$$U(t) \approx 1 + \sum_{n \geq 1} \frac{(-i)^n}{n!}\int d^4x_1 \int d^4x_2 \ldots \int d^4x_n T\{I(x_1)\ldots I(x_n)\} \tag{E1.2}$$

Appendix F Locality of Dirac Field Operators

Theorem: The equal time anticommutation relations for the Dirac field and Dirac adjoint and obey:

$$\{\psi_\alpha(x), \widehat{\psi}_\beta(y)\}_{x^0 = y^0} = \gamma^0_{\alpha\beta}\delta(x - y) \tag{F1.1}$$

Proof: Using the identity (true in a particular basis, so true in any basis),

$$\sum_r \zeta(r)\zeta(r)^\dagger = 1_2$$

We have $(\boldsymbol{\sigma} \cdot \boldsymbol{p})^2 = ((p^0)^2 - m^2)1_2$ (C1.2). Thus,

$$\sum_r u_\alpha(\boldsymbol{p}, r)\widehat{u}_\beta(\boldsymbol{p}, r) = \frac{p^0 + m}{2p_0}\sum_r \begin{bmatrix} \zeta(r) \\ \dfrac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{p^0 + m}\zeta(r) \end{bmatrix} \begin{bmatrix} \zeta(r)^\dagger & -\zeta(r)^\dagger \dfrac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{p^0 + m}^\dagger \end{bmatrix}$$

$$= \frac{1}{2p_0}\begin{bmatrix} (p^0 + m)1_2 & -\boldsymbol{\sigma} \cdot \boldsymbol{p}^\dagger \\ \boldsymbol{\sigma} \cdot \boldsymbol{p} & -\dfrac{(\boldsymbol{\sigma} \cdot \boldsymbol{p})^2}{p^0 + m}1_2 \end{bmatrix}$$

$$= \frac{1}{2p_0}\begin{bmatrix} (p^0 + m)1_2 & -\boldsymbol{\sigma} \cdot \boldsymbol{p} \\ \boldsymbol{\sigma} \cdot \boldsymbol{p} & -(p^0 - m)1_2 \end{bmatrix}$$

$$= \frac{1}{2p^0}(p \cdot \gamma + m) \,.$$

Similarly,

$$\sum_r v_\alpha(\boldsymbol{p}, r)\widehat{v}_\beta(\boldsymbol{p}, r) = \frac{1}{2p^0}(p \cdot \gamma - m) \,.$$

We have

$$\{\psi_\alpha(x), \widehat{\Psi}_\beta(y)\} = \{\langle \underline{x}, \alpha|, |\widehat{\underline{y}}, \beta\rangle\} + \{|\overline{x}, \alpha\rangle, \langle\widehat{\overline{y}}, \beta|\} = \langle \underline{x}, \alpha|\widehat{\underline{y}}, \beta\rangle + \langle\widehat{\overline{y}}, \beta|\overline{x}, \alpha\rangle^{\mathrm{T}}$$

where T denotes that α and β are transposed. Using the resolution of unity and the solution of the Dirac equation,

$$\langle \underline{x}, \alpha | \widehat{\underline{y}}, \beta \rangle = \frac{1}{8\pi^3} \sum_r \int d^3\boldsymbol{p}\, u_\alpha(\boldsymbol{p}, r) \widehat{u}_\alpha(\boldsymbol{p}, r) e^{-ip \cdot (x-y)}$$

$$= \frac{1}{8\pi^3} \int \frac{d^3\boldsymbol{p}}{2p^0} (p \cdot \gamma + m)_{\alpha\beta} e^{-ip \cdot (x-y)} \;. \tag{F1.2}$$

Likewise for the antiparticle,

$$\langle \widehat{\overline{y}}, \beta | \overline{x}, \alpha \rangle^{\mathrm{T}} = \frac{1}{8\pi^3} \sum_r \int d^3\boldsymbol{p}\; v_\alpha(\boldsymbol{p}, r) \widehat{v}_\beta\; (\boldsymbol{p}, r)\; e^{ip \cdot y - ix \cdot p}$$

$$= \frac{1}{8\pi^3} \int \frac{d^3\boldsymbol{p}}{2p^0}\; (p \cdot \gamma - m)_{\alpha\beta}\; e^{ip \cdot (x-y)} \;. \tag{F1.3}$$

Substituting $\boldsymbol{p} \to -\boldsymbol{p}$ at $x_0 = y_0$

$$\langle \widehat{\overline{y}}, \beta | \overline{x}, \alpha \rangle^{\mathrm{T}} = \frac{1}{8\pi^3} \int \frac{d^3\boldsymbol{p}}{2p^0}\; (2p_0\gamma^0 - p \cdot \gamma - m)_{\alpha\beta} e^{-ip \cdot (x-y)} \tag{F1.4}$$

Adding at $x^0 = y^0$ gives the equal time anticommutator,

$$\{\psi_\alpha(x), \widehat{\Psi}_\beta(y)\}_{x^0 = y^0} = \frac{1}{8\pi^3} \gamma^0_{\alpha\beta} \int d^3\boldsymbol{p}\; e^{-ip \cdot (x-y)} = \gamma^0_{\alpha\beta} \delta(\boldsymbol{x} - \boldsymbol{y}) \,,$$

as required.

Theorem: The anticommutation relation for the Dirac field and the Dirac adjoint is zero outside the light cone.

Proof: From the above,

$$\langle \underline{x} | \widehat{\underline{y}} \rangle = \frac{1}{8\pi^3} (i\partial \cdot \gamma + m) \int \frac{d^3\boldsymbol{p}}{2p^0} e^{-ip \cdot (x-y)} \tag{F1.5}$$

and

$$\langle \widehat{\overline{y}} | \overline{x} \rangle^{\mathrm{T}} = -\frac{1}{8\pi^3} (i\partial \cdot \gamma + m) \int \frac{d^3\boldsymbol{p}}{2p^0}\; e^{ip \cdot (x-y)} \,. \tag{F1.6}$$

The anticommutator is found by adding:

$$\{\psi(x), \widehat{\Psi}(y)\} = \frac{1}{8\pi^3} (i\partial \cdot \gamma + m) \int \frac{d^3\boldsymbol{p}}{2p^0} (e^{-ip \cdot (x-y)} - e^{ip \cdot (x-y)})$$

$$= \frac{1}{8\pi^3} (i\partial \cdot \gamma + m) \int d^4p (e^{-ip \cdot (x-y)} - e^{ip \cdot (x-y)}) \delta(p^2 - m^2) \,,$$

using the generalised scaling property of the delta function applied to the mass shell condition. The integral is Lorentz invariant and is zero when $x^0 - y^0 = 0$. We conclude that it is zero whenever $x - y$ is spacelike.

Appendix G Gauge Invariance

The local phase transformation,

$$\psi(x) \to e^{i\alpha(x)} \psi(x) \tag{G1.1}$$

applied to the field operators, makes no difference to the current and so leaves the predictions of the theory unchanged (equivalently the transformation may be applied to the creation operators, remembering the sign change for antiparticles). The interacting Dirac equation,

$$(\gamma^a(i\partial_a - e\cdot\langle A_a(x)\rangle) - m)\,f(x) = 0\,, \tag{G1.2}$$

can be written, in terms of creation operators acting on any ket $|f\rangle$,

$$\int d^3x|\underline{x}\rangle(\gamma^a(i\partial_a - e\cdot\langle A_a(x)\rangle) - m)\langle\underline{x}||f\rangle = 0\,. \tag{G1.3}$$

A local gauge transformation applied to the creation operators,

$$|\underline{x}\rangle \rightarrow e^{-i\alpha(x)}|\underline{x}\rangle\,, \tag{G1.4}$$

gives

$$\int d^3x|\underline{x}\rangle e^{-i\alpha(x)}(\gamma^a(i\partial_a - e\cdot\langle A_a(x)\rangle) - m)e^{i\alpha(x)}\langle\underline{x}||f\rangle = 0\,. \tag{G1.5}$$

So,

$$\int d^3x|\underline{x}\rangle(\gamma^a(i\partial_a - \partial_a\alpha(x) - e\cdot\langle A_a(x)\rangle) - m)e^{i\alpha(x)}\langle\underline{x}||f\rangle = 0\,.$$

$$\int d^3x|\underline{x}\rangle(\gamma^a(i\partial_a - e\cdot\langle A_a(x)\rangle') - m)\langle\underline{x}||f\rangle = 0\,. \tag{G1.6}$$

which is identical to the original form of the interacting Dirac equation apart from the replacement

$$\langle A_a(x)\rangle' = \langle A_a(x)\rangle + \frac{1}{e}\partial_a\alpha(x)\,. \tag{G1.7}$$

But the Faraday tensor is also unchanged by this replacement;

$$F^{ab} = \partial^a\langle A^b(x)\rangle - \partial^b\langle A^a(x)\rangle = \partial^a\langle A^b(x)\rangle' - \partial^b\langle A^a(x)\rangle'\,. \tag{G1.8}$$

So the local phase symmetry of the field operators is precisely equivalent to the well known symmetry of the classical electromagnetic field.

Appendix H Feynman Diagrams

H1 The Time Ordered Vertex for QED

The interaction density for qed is

$$I(x) = ej^a(x)A_a(x) = e(|\widehat{\underline{x}}\rangle + \langle\widehat{\overline{x}}|)\gamma^a(|\overline{x}\rangle + \langle\underline{x}|)(|\underline{x},a\rangle + \langle\underline{x},a|)$$

where photon creation and annihilation operators are distinguished by the vector index, a. $I(x)$ is the sum of eight terms, each of which can be represented can be represented diagrammatically as a time ordered vertex or node (figure 3). Lines above the node correspond to creation operators, and those below the node correspond to annihilation operators. The photon is represented by a wavy line, electrons by a upward arrow and positrons by a downward arrow.

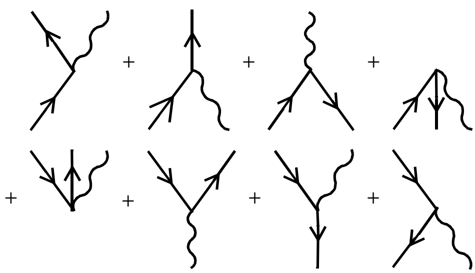

Figure 3: Time-ordered vertex for qed

H2 Wick's Theorem

Wick's theorem can be used to replace the time ordered product with a normal ordered product by (anti)commuting annihilation operators to the right and creation operators to the left. Let ϕ be the field operator,

$$\phi(x) = |\overline{x}\rangle + \langle \underline{x}|$$

If x is before y, $x^0 < y^0$, the Feynman propagator, $D(x - y)$, gives the amplitude for the creation of an antiparticle at x and its annihilation at y. If x is after y, $x^0 > y^0$ it gives the amplitude for creation of a particle at y and its annihilation at x.

Theorem: (Wick's Theorem) For two field operators,

$$T\{\phi^\dagger(x)\phi(y)\} = :\phi^\dagger(x)\phi(y): + D(x-y)\,. \tag{H2.1}$$

For n field operators:

$$T\{\phi^\dagger(x_1)\ldots\phi^\dagger(x_i)\phi(x_{i+1})\ldots\phi(x_n)\}$$
$$= :\phi^\dagger(x_1)\ldots\phi^\dagger(x_i)\phi(x_{i+1})\ldots\phi(x_n): + \sum_{\substack{\text{all pairs of}\\ \text{contractions}}} :\phi^\dagger(x_1)\ldots\phi(x_n): \prod_{\text{pairs } j,\,k} D(x_j - x_k)$$

where $1 = i, j, k = n$ and contracted pairs are omitted in the normal ordered product.

Proof: A detailed proof by induction can be carried out, but the proof is no more evident than the theorem itself, which just means that we do the normal ordering by carrying out the contractions.

H3 The S-matrix

Initial and final states can be expressed as sums of plane wave states by using the resolution of unity in momentum space. The time evolution between t_0 and t_1 is given by a matrix in momentum space

$$\langle p_1;\ldots;p_j|U(t_1, t_0)|p_{j+1};\ldots;p_n\rangle$$

In the case of scattering, the initial state (generated by a particle accelerator), and the final state (typically measured by bubble chamber, wire chamber or silicon detector) are well represented as pure momentum states. In this case the interesting interaction takes place at the scattering event, and t_0 and t_1 are not important.

Postulate: The ***S*-matrix** (or **scattering matrix**) is

$$\langle p_1;\ldots;p_j|S|p_j;\ldots;p_n\rangle = \lim_{\substack{t_0\to-\infty\\ t_1\to\infty}} \langle p_1;\ldots;p_j|U(t_1, t_0)|p_{j+1};\ldots;p_n\rangle \tag{H3.1}$$

The S-matrix is found from the perturbation expansion by first normal ordering the terms using Wick's theorem. Then, for the interaction density at x, the creation operator acting on the initial state $|\boldsymbol{p}, r\rangle$ gives, for a photon,

$$\langle \underline{x}|\boldsymbol{p}, r\rangle = \left(\frac{1}{2\pi}\right)^{3/2} \frac{w(\boldsymbol{p}, r)}{\sqrt{2p^0}} e^{-ip\cdot x},$$

for a Dirac particle,

$$\langle\, \underline{x} | \boldsymbol{p}, r \,\rangle \;=\; \left(\tfrac{1}{2\pi}\right)^{3/2} u(\boldsymbol{p}, r) e^{-ip \cdot x}\,,$$

and for an antiparticle,

$$\langle \overset{\frown}{\overline{x}} | \boldsymbol{p}, r \,\rangle \;=\; \left(\tfrac{1}{2\pi}\right)^{3/2} \overset{\frown}{v}(\boldsymbol{p}, r) e^{-ip \cdot x}\,.$$

Similarly, the annihilation operators in the interaction density acting on the final state $\langle\, \boldsymbol{p}, r \,|$ gives, for a photon,

$$\langle\, \boldsymbol{p}, r | \underline{x} \rangle \;=\; \left(\tfrac{1}{2\pi}\right)^{3/2} \frac{w(p, r)}{\sqrt{2p^0}} e^{ip \cdot x}\,,$$

for a Dirac particle,

$$\langle\, \boldsymbol{p}, r | \overset{\frown}{\underline{x}} \rangle \;=\; \left(\tfrac{1}{2\pi}\right)^{3/2} \overset{\frown}{u}(\boldsymbol{p}, r) e^{ip \cdot x}\,,$$

and for an antiparticle,

$$\langle\, \boldsymbol{p}, r | \overline{x} \rangle \;=\; \left(\tfrac{1}{2\pi}\right)^{3/2} v(\boldsymbol{p}, r) e^{ip \cdot x}\,.$$

To keep track of the contractions in normal ordering the perturbation expansion, the terms are represented by graphs. A particle created at x^0 may be annihilated at a later time y^0. An antiparticle created at x^0 may be annihilated at an earlier time y^0. Each contraction is represented by connecting the corresponding lines between vertices, and, at the same time, removing time ordering (figure 4).

After carrying out the contractions, all topologically equivalent time ordered diagrams are combined into a single diagram with no time ordering between the nodes (figure 5). There are $k!$ diagrams with n nodes. So, removing the ordering of nodes generates a factor $k!$ and cancels the factor $1/k!$ in the perturbation expansion (3.5.2), leaving a sum for a diagram with n vertices,

$$U(t_n) \;=\; (-ie)^k \gamma^{a_1}\gamma^{a_2}\ldots\gamma^{a_n}(-i\chi^4)^k \sum_{i_k \neq i_{k-1},\, i_k,\, \ldots,\, i_1} \cdots \sum_{i_2 \neq i_1} \sum_{i_1 = 1}^{n}\,. \tag{H3.2}$$

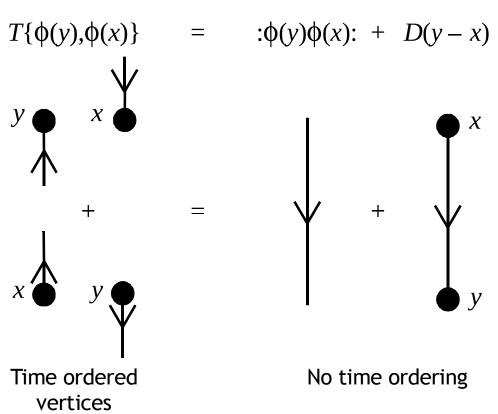

Figure 4: Contraction represented by connecting vertices and removing time ordering

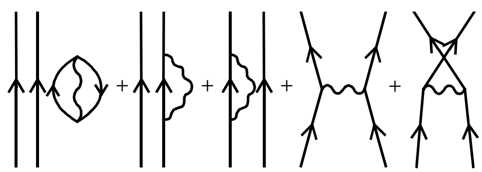

Figure 5a: Second order diagrams for initial and final states with two particles. The final term is zero if the particles can be distinguished (e.g. if one is bound to an atom).

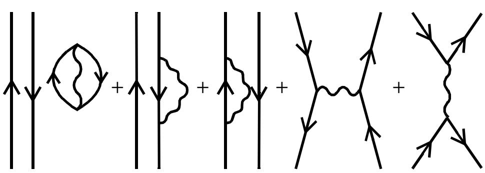

Figure 5b: Second order diagrams for initial and final states with a particle and an antiparticle.

H4 Conservation of Energy and Momentum

Gather all the exponential terms from internal and external lines with x_i in the exponent. Provided the time from t_0 to t_n is large, the result is a delta function

$$\left(\frac{\chi}{2\pi}\right)^4 \sum_S e^{-i\tilde{p}x_i + i\tilde{q}x_i - i\tilde{k}x_i} = \delta^{(4)}(\tilde{p} - \tilde{q} + \tilde{k})$$

where $\tilde{p}, \tilde{q}, \tilde{k}$ refer to the arrowed line coming from vertex, the arrowed line going into the vertex, and the photon line, respectively. The delta function shows that the tilda'd quantities are conserved. For internal lines, $\tilde{p}^0, \tilde{q}^0, \tilde{k}^0$, are the dummy variables introduced in the contour integration. For external lines $(\tilde{p}^0, \tilde{q}^0, \tilde{k}^0) = (p^0, q^0, k^0)$. Energy, p^0, was originally defined to be the zero component of a vector. This is not a conserved quantity. Vectors are products of measurement, and only have real meaning in measurement. By definition, internal lines do not correspond to measured states. So, p^0 has no meaning on internal lines in a Feynman diagram. The conserved tilda'd quantities are of more interest than vector quantities and it is usual to *redefine* energy.

Redefinition: *Energy* is the conserved quantity, $\tilde{p}^0$, which appears on the lines of a Feynman diagram.

With this definition, energy-momentum, $\tilde{p}$, is conserved, but is not a vector, and does not obey the mass shell condition on internal lines in Feynman diagrams. Particles are said to be off shell on internal lines. On external lines, representing measured states, this definition of energy coincides with the original definition for measured states, as the time component of a vector. Particles are said to be on shell on external lines, meaning that the mass shell condition is obeyed in measurement.

H5 Feynman Rules

After using the delta functions to carry out the integrals over tilda'd quantities, and imposing the rule that energy-momentum is conserved at each vertex, there remains an integral for each independent internal loop,

$$\frac{1}{16\pi^4}\int d^4\tilde{p}\,.$$

Each vertex contributes a factor

$$-ie\gamma^a\,.$$

For external lines in the initial state we have, for a photon,

$$\sqrt{\frac{1}{4\pi p^0}}\,w(\boldsymbol{p}, r)\,,$$

for a Dirac particle,

$$\sqrt{\frac{1}{2\pi}}\,u(p, r)\,,$$

and for an antiparticle,

$$\sqrt{\frac{1}{2\pi}}\,\overset{\frown}{v}(\boldsymbol{p}, r)\,.$$

For external lines in the final state we have, for a photon,

$$\sqrt{\frac{1}{4\pi p^0}}w(p, r)\,,$$

for a Dirac particle,

$$\sqrt{\frac{1}{2\pi}}\widehat{u}(\boldsymbol{p}, r)\,,$$

and for an antiparticle,

$$\sqrt{\frac{1}{2\pi}}v(\boldsymbol{p}, r)\,.$$

For internal arrowed lines we have

$$\frac{ip\cdot\gamma + m}{\tilde{p}^2 - m^2 + i\varepsilon}\,,$$

and for internal photon lines we have

$$\frac{-ig_{ab}}{\tilde{p}^2 + 2i|\boldsymbol{p}|\varepsilon + \varepsilon^2}\,.$$

In addition there is a minus sign if an odd number of commutations of Fermion creation and annihilation operators is required to put the diagram into normal order. The limit $\varepsilon \to 0$ should be taken after evaluation of integrals for loops and for the initial and final states. If $|\boldsymbol{p}| > 0$ then the photon propagator can be replaced with

$$\frac{-ig_{ab}}{\tilde{p}^2 + i\varepsilon}\,.$$

Certain diagrams contain a divergence when photon energy goes to zero. In this case ε^2 should be retained until after evaluation of the integral to control the infrared divergence (ε^2 plays the role of the small photon mass commonly used for this purpose).

Appendix I Derivation of Propagators

I1 Lemma

$$\lim_{\varepsilon\to 0^+}\int_{-\infty}^{\infty} d\tilde{p}^0 \frac{e^{-i\tilde{p}^0x^0}}{\tilde{p}^2 - m^2 + 2ip^0\varepsilon + \varepsilon^2} = \begin{cases} -2\pi i\dfrac{e^{-ip^0x^0}}{2p^0} & \text{if } x^0 > 0. \\ -2\pi i\dfrac{e^{ip^0x^0}}{2p^0} & \text{if } x^0 < 0. \end{cases} \tag{I1.1}$$

Proof: Since $\tilde{p} = (\tilde{p}^0, \pm\boldsymbol{p})$,

$$\tilde{p}^2 - m^2 = \tilde{p}^2 - p^2 = (\tilde{p}^0)^2 - (p^0)^2$$

So,

$$\int_{-\infty}^{\infty} d\tilde{p}^0 \frac{e^{-i\tilde{p}^0x^0}}{\tilde{p}^2 - m^2 + 2ip^0\varepsilon + \varepsilon^2} = \int_{-\infty}^{\infty} d\tilde{p}^0 \frac{e^{-i\tilde{p}^0x^0}}{(\tilde{p}^0)^2 - (p^0 - i\varepsilon)^2}$$

$$= \int_{-\infty}^{\infty} d\tilde{p}^0 \frac{e^{-i\tilde{p}^0x^0}}{(\tilde{p}^0 - p^0 + i\varepsilon)(\tilde{p}^0 + p^0 - i\varepsilon)}\,.$$

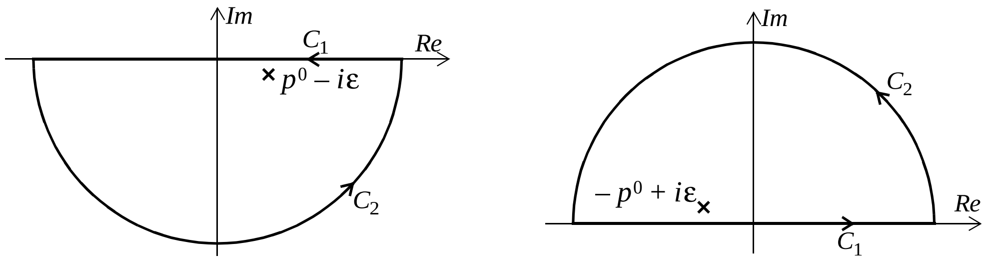

Figure 6: Contours for the integrations in the complex $\tilde{p}^0$ plane.

This is evaluated as a contour integral, and noting that the integral on C_2 vanishes in the lower half plane if $x^0 > 0$, and in the upper half plane if $x^0 < 0$ (figure 6).

I2 The Photon Propagator

Using the lemma (I1.1)

$$D_F(x_i - x_j) = \Theta(x_i^0 - x_j^0)\langle \underline{x}_i | \underline{x}_j \rangle + \Theta(x_j^0 - x_i^0)\langle \underline{x}_j | \underline{x}_i \rangle^{\mathrm{T}}$$

$$= \frac{-g}{8\pi^3}\int d^3\boldsymbol{p}\left[\Theta(x_i^0 - x_j^0)\frac{e^{-ip\cdot(x_j - x_i)}}{2p^0} + \Theta(x_j^0 - x_i^0)\frac{e^{ip\cdot(x_j - x_i)}}{2p^0}\right]$$

$$= \frac{-ig}{16\pi^4}\lim_{\varepsilon\to 0^+}\int d^3\boldsymbol{p}\int_{-\infty}^{\infty} d\tilde{p}^0[\Theta(x_i^0 - x_j^0) + \Theta(x_j^0 - x_i^0)]\frac{e^{-i\tilde{p}\cdot(x_j - x_i)}}{\tilde{p}^2 + 2ip^0\varepsilon + \varepsilon^2}$$

where $\tilde{p} = (\tilde{p}^0, \boldsymbol{p})$ if $x_i^0 > x_j^0$ and $\tilde{p} = (\tilde{p}^0, -\boldsymbol{p})$ if $x_j^0 > x_i^0$ is the momentum from j to i. If $x^0 \neq y^0$, the step functions can be summed to unity,

$$D_F(x_i - x_j) = \frac{-ig}{16\pi^4}\lim_{\varepsilon\to 0^+}\int d^4\boldsymbol{p}\frac{e^{-i\tilde{p}\cdot(x_j - x_i)}}{\tilde{p}^2 + 2ip^0\varepsilon + \varepsilon^2} \tag{I2.1}$$

I3 The Dirac Propagator

It has been shown that (F1.5)

$$\langle \underline{x} | \overset{\frown}{\underline{y}} \rangle = \frac{1}{8\pi^3}(i\partial \cdot \gamma + m)\int \frac{d^3\boldsymbol{p}}{2p^0}e^{-ip\cdot(x - y)}$$

and (F1.6)

$$\langle \overset{\frown}{\overline{y}} | \overline{x} \rangle^{\mathrm{T}} = -\tfrac{1}{8\pi^3}(i\partial \cdot \gamma + m)\int \frac{d^3\boldsymbol{p}}{2p^0}\; e^{ip\cdot(x - y)}.$$

Hence, by the lemma (I1.1),

$$S_F(x_i - x_j) = \Theta(x_i^0 - x_j^0)\langle \underline{x}_i | \overset{\frown}{\underline{x}}_j \rangle - \Theta(x_j^0 - x_i^0)\langle \overset{\frown}{\overline{x}}_j | \overline{x}_i \rangle^{\mathrm{T}}$$

$$= \frac{(i\partial \cdot \gamma + m)}{8\pi^3}\int \frac{d^3\boldsymbol{p}}{2p^0}[\Theta(x_i^0 - x_j^0)e^{-ip\cdot(x_i - x_j)} + \Theta(x_j^0 - x_i^0)e^{ip\cdot(x_i - x_j)}]$$

$$= i\lim_{\varepsilon\to 0^+}\frac{(i\partial \cdot \gamma + m)}{16\pi^4}\int d^3\tilde{\boldsymbol{p}}\int_{-\infty}^{\infty} d\tilde{p}^0[\Theta(x_i^0 - x_j^0) + \Theta(x_j^0 - x_i^0)]\frac{e^{-ip\cdot(x_i - x_j)}}{\tilde{p}^2 - m^2 + 2ip_0\varepsilon + \varepsilon^2},$$

where $\tilde{p} = (\tilde{p}^0, \boldsymbol{p})$ if $x_i^0 > x_j^0$ and $\tilde{p} = (p^0, -\boldsymbol{p})$ if $x_j^0 > x_i^0$ is the momentum from j to i (in the direction of the arrow). If $x^0 \neq y^0$, the step functions can be summed to unity,

$$S_F(x_i - x_j) = i \lim_{\varepsilon \to 0^+} \frac{(i\partial \cdot \gamma + m)}{16\pi^4} \int d^4\tilde{\boldsymbol{p}} \frac{e^{-i\tilde{p} \cdot (x_i - x_j)}}{\tilde{p}^2 - m^2 + 2ip^0\varepsilon + \varepsilon^2}.$$

For a Dirac particle, $p^0 > 0$, and we can simplify the denominator by shifting the pole under the limit, replacing $2ip^0\varepsilon + \varepsilon^2$ with $i\varepsilon$. Thus the Dirac propagator arrowed from j to i is

$$S_F(x_i - x_j) = \frac{i}{16\pi^4} \int d^4\tilde{\boldsymbol{p}} \frac{(\tilde{p} \cdot \gamma + m)e^{-i\tilde{p} \cdot (x_i - x_j)}}{\tilde{p}^2 - m^2 + i\varepsilon}. \quad \text{(I3.1)}$$

Appendix J The Origin of the Ultraviolet Divergence

It is well known that an integral which contains a squared delta-function in the integrand,

$$S = \int_a^b \delta^2(x - h)dx, \quad \text{(J1.1)}$$

cannot be defined for $a < h < b$. The origin of the ultraviolet divergence has been identified as the inclusion of such terms in Feynman diagrams containing loop integrals. Now consider the improper integral with $\varepsilon > 0$

$$S' = \lim_{\varepsilon \to 0} \int_a^{h-\varepsilon} \delta^2(x - h)dx + \lim_{\varepsilon \to 0} \int_{h+\varepsilon}^b \delta^2(x - h)dx. \quad \text{(J1.2)}$$

S' is (or at least, can be) well defined and trivially evaluates to zero. When the order of taking limits is properly tracked, the origin of the ultraviolet divergence is seen in the replacement of well defined integrals containing terms of the form S' with undefined integrals containing terms of the form S.

The usual method of subtracting divergent quantities from loop diagrams is then seen as equivalent to subtracting a term,

$$S'' = S - S'. \quad \text{(J1.3)}$$

This restores the correct answer, but it means working with undefined quantities and the usual rationale is wrong. No renormalisation is involved, and nor does the subtraction require adding counterterms to the Hamiltonian, because when the order of taking limits is tracked the divergence is not present in the original form of the perturbation expansion.

The appearance of squared delta functions in the integrand can be traced to the equal point multiplication between fields, which cannot be defined when fields are operator valued distributions. However, there is no equal point multiplication in (E1.1), because all inequalities in the bounds of integration are strict. The equal point multiplication appears in (E1.2) as a consequence of incorrectly changing the order of taking limits. The exclusion of the equal point multiplication can also be seen as a physical constraint, that an electron cannot interact more than once in any instant. This statement is given a clear physical meaning through the introduction of a minimum discrete unit of time, χ.